UNIT – III

STEADY STATE ANALYSIS OF A.C CIRCUITS

Introduction

The resistance, inductance and capacitance are three basic elements of any electrical network. In order to analyze any electric circuit, it is necessary to understand the following three cases,

1) A.C. through pure resistive circuit.
2) A.C. through pure inductive circuit.
3) A.C. through pure capacitive circuit.

In each case, it is assumed that a purely sinusoidal alternating voltage given by the equation $v = V_m \sin(\omega t)$ is applied to the circuit. The equation for the current, power and phase shift are developed in each case.

A.C. Through Pure Resistance

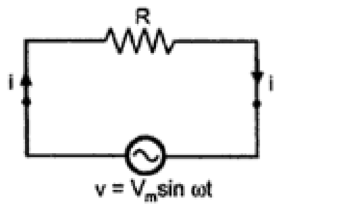


Fig. 4.1 Pure resistive circuit

Consider a simple circuit consisting of a pure resistance 'R' ohms connected across a voltage $v = V_m \sin \omega t$. The circuit is shown in the Fig. 4.1.

According to Ohm's law, we can find the equation for the current i as,

$$i = \frac{v}{R} = \frac{V_m \sin \omega t}{R}$$

i.e. $$i = \left(\frac{V_m}{R}\right) \sin(\omega t)$$

This is the equation giving instantaneous value of the current.

Comparing this with standard equation,

$$i = I_m \sin(\omega t + \phi)$$

$$\boxed{I_m = \frac{V_m}{R} \quad \text{and} \quad \phi = 0}$$

So, maximum value of alternating current, i is $I_m = \frac{V_m}{R}$ while, as $\phi = 0$, it indicates that it is in phase with the voltage applied. There is no phase difference between the two. The current is going to achieve its maximum (positive and negative) and zero whenever voltage is going to achieve its maximum (positive and negative) and zero values.

In purely resistive circuit, the current and the voltage applied are in phase with each other.

The waveforms of voltage and current and the corresponding phasor diagram is shown in the Fig. 4.2 (a) and (b).

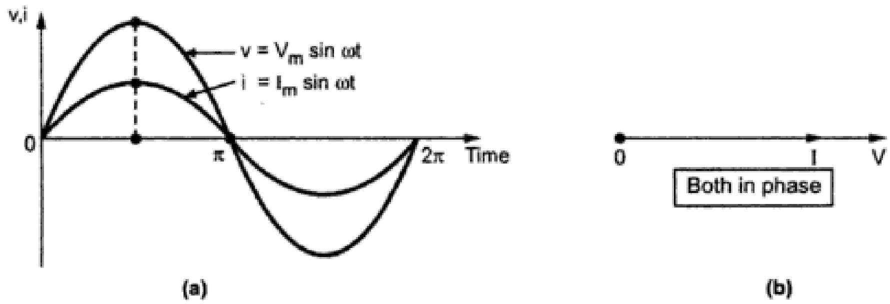


Fig. 4.2 A.C. through purely resistive circuit

In the phasor diagram, the phasors are drawn in phase and there is no phase difference in between them. Phasors represent the **r.m.s. values** of alternating quantities.

Power

The instantaneous power in a.c. circuits can be obtained by taking product of the instantaneous values of current and voltage.

$$P = v \times I$$

$$= V_m \sin(\omega t) \times I_m \sin \omega t$$

$$= V_m I_m \sin^2(\omega t)$$

$$= \frac{V_m I_m}{2}(1 - \cos 2\omega t)$$

$$\therefore \quad \boxed{P = \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos(2\omega t)}$$

From the above equation, it is clear that the instantaneous power consists of two components,

1) Constant power component $\left(\frac{V_m I_m}{2}\right)$

2) Fluctuating component $\left[\frac{V_m I_m}{2} \cos(2\omega t)\right]$ having frequency, double the frequency of the applied voltage.

Now, the average value of the fluctuating cosine component of double frequency is zero, over one complete cycle. So, average power consumption over one cycle is equal to the constant power component i.e. $\frac{V_m I_m}{2}$.

$$\therefore \quad P_{av} = \frac{V_m I_m}{2} = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}}$$

$$\therefore \quad \boxed{P_{av} = V_{rms} \times I_{rms} \quad \text{watts}}$$

Generally, r.m.s. values are indicated by capital letters

$$\therefore \quad \boxed{P_{av} = V \times I \quad \text{watts} = I^2 R \quad \text{watts}}$$

The Fig. 4.3 shows the waveforms of voltage, current and power.

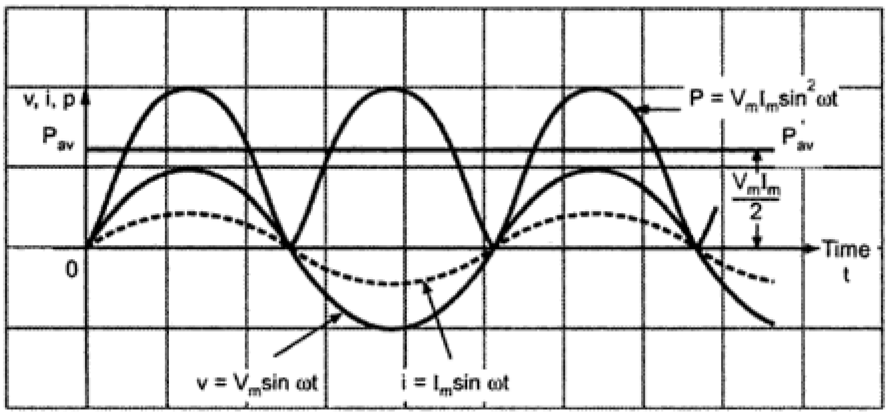


Fig. 4.3 v, i and p for purely resistive circuit

A.C. Through Pure Inductance

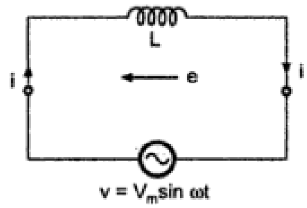


Fig. 4.4 Purely inductive circuit

Consider a simple circuit consisting of a pure inductance of L henries, connected across a voltage given by the equation, $v = V_m \sin \omega t$. The circuit is shown in the Fig. 4.4.

Pure inductance has zero ohmic resistance. Its internal resistance is zero. The coil has pure inductance of L henries (H).

When alternating current 'i' flows through inductance 'L', it sets up an alternating magnetic field around the inductance. This changing flux links the coil and due to self inductance, e.m.f. gets induced in the coil. This e.m.f. opposes the applied voltage.

The self induced e.m.f. in the coil is given by,

Self induced e.m.f., $e = -L\dfrac{di}{dt}$

At all instants, applied voltage, v is equal and opposite to the self induced e.m.f., e

$$\therefore \quad v = -e = -\left(-L\frac{di}{dt}\right)$$

$$\therefore \quad v = L\frac{di}{dt}$$

$$\therefore \quad V_m \sin \omega t = L\frac{di}{dt}$$

$$\therefore \quad di = \frac{V_m}{L}\sin \omega t \, dt$$

$$\therefore \quad i = \int di = \int \frac{V_m}{L}\sin \omega t \, dt = \frac{V_m}{L}\left(\frac{-\cos \omega t}{\omega}\right)$$

$$= -\frac{V_m}{\omega L}\sin\left(\frac{\pi}{2}-\omega t\right) \quad \text{as } \cos \omega t = \sin\left(\frac{\pi}{2}-\omega t\right)$$

$$\therefore \quad i = \frac{V_m}{\omega L}\sin\left(\omega t-\frac{\pi}{2}\right) \quad \text{as } \sin\left(\frac{\pi}{2}-\omega t\right) = -\sin\left(\omega t-\frac{\pi}{2}\right)$$

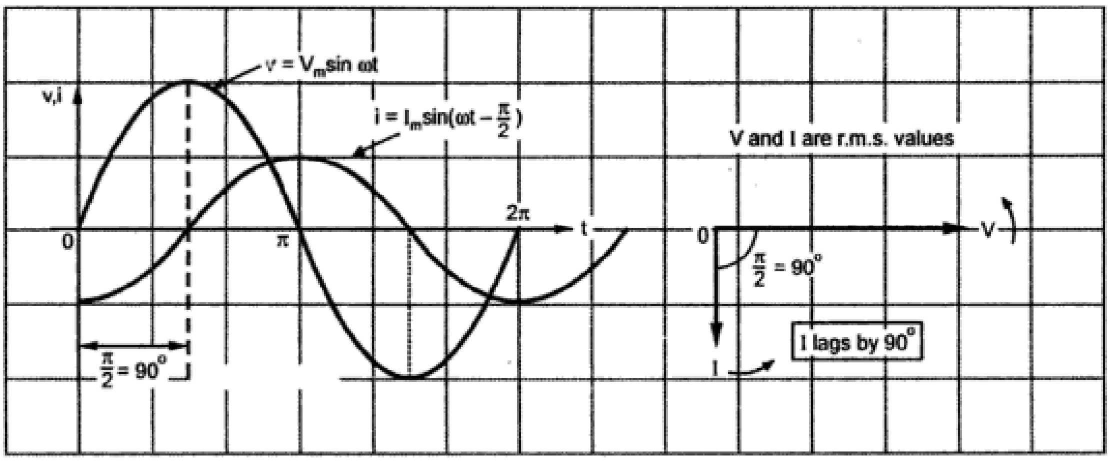
$$\therefore \quad \boxed{i = I_m \sin\left(\omega t-\frac{\pi}{2}\right)}$$

where $I_m = \dfrac{V_m}{\omega L} = \dfrac{V_m}{X_L}$

where $\boxed{X_L = \omega L = 2\pi f L \ \Omega}$

The above equation clearly shows that the current is purely sinusoidal and having phase angle of $-\frac{\pi}{2}$ radians i.e. $-90°$. This means that **the current lags voltage applied by 90°**. The negative sign indicates lagging nature of the current. If current is assumed as a reference, we can say that the voltage across inductance leads the current passing through the inductance by 90°.

The Fig. 4.5 shows the waveforms and the corresponding phasor diagram



(a) Waveforms (b) Phasor diagram

Fig. 4.5 A.C. through purely inductive circuit

In purely inductive circuit, current lags voltage by 90°.

Concept of Inductive Reactance

We have seen that in purely inductive circuit,

$$I_m = \frac{V_m}{X_L}$$

where $\boxed{X_L = \omega L = 2\pi f L\ \Omega}$

The term, X_L, is called **Inductive Reactance** and is measured in **ohms.**

So, **inductive reactance** is defined as the opposition offered by the inductance of a circuit to the flow of an alternating sinusoidal current.

It is measured in ohms and it depends on the frequency of the applied voltage.

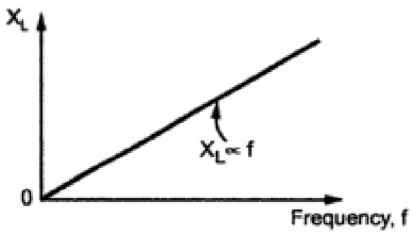


Fig. 4.6 X_L Vs f

The inductive reactance is directly proportional to the frequency for constant L.

$X_L \propto f$, for constant L

So, graph of X_L Vs f is a straight line passing through the origin as shown in the Fig. 4.6.

Key Point : *If frequency is zero, which is so for d.c. voltage, the inductive reactance is zero. Therefore, it is said that the inductance offers zero reactance for the d.c. or steady current.*

Power The expression for the instantaneous power can be obtained by taking the product of instantaneous voltage and current.

$$\therefore \quad P = v \times i$$

$$= V_m \sin \omega t \times I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= -V_m I_m \sin(\omega t)\cos(\omega t) \quad \text{as } \sin\left(\omega t - \frac{\pi}{2}\right) = -\cos \omega t$$

$$\therefore \quad \boxed{P = -\frac{V_m I_m}{2} \sin(2\omega t)} \quad \text{as } 2\sin \omega t \cos \omega t = \sin 2\omega t$$

This power curve is a sine curve of frequency double than that of applied voltage.

The average value of sine curve over a complete cycle is always zero.

$$P_{av} = \int_0^{2\pi} -\frac{V_m I_m}{2} \sin(2\omega t)\, d(\omega t) = 0$$

The Fig. 4.7 shows voltage, current and power waveforms.

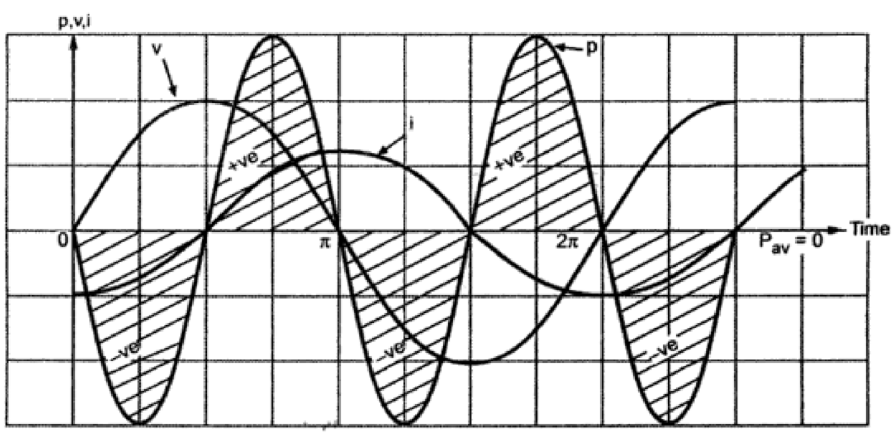


Fig. 4.7 Waveforms of voltage, current and power

when power curve is positive, energy gets stored in the magnetic field established due to the increasing current while during negative power curve, this power is returned back to the supply.

The areas of positive loop and negative loop are exactly same and hence, average power consumption is zero.

Pure inductance never consumes power.

A.C. Through Pure Capacitance

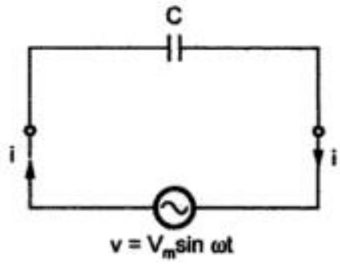


Fig. 4.8 Purely capacitive circuit

Consider a simple circuit consisting of a pure capacitor of C-farads, connected across a voltage given by the equation, $v = V_m \sin \omega t$. The circuit is shown in the Fig. 4.8.

The current i charges the capacitor C. The instantaneous charge 'q' on the plates of the capacitor is given by,

$$q = C\,v$$

$$\therefore \quad q = C\,V_m \sin \omega t$$

Now, current is rate of flow of charge.

$$\therefore \qquad i = \frac{dq}{dt} = \frac{d}{dt}(C\,V_m \sin \omega t)$$

$$\therefore \qquad i = C\,V_m \frac{d}{dt}(\sin \omega t) = C\,V_m\,\omega \cos \omega t$$

$$\therefore \qquad i = \frac{V_m}{\left(\frac{1}{\omega C}\right)} \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$\therefore \qquad i = I_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

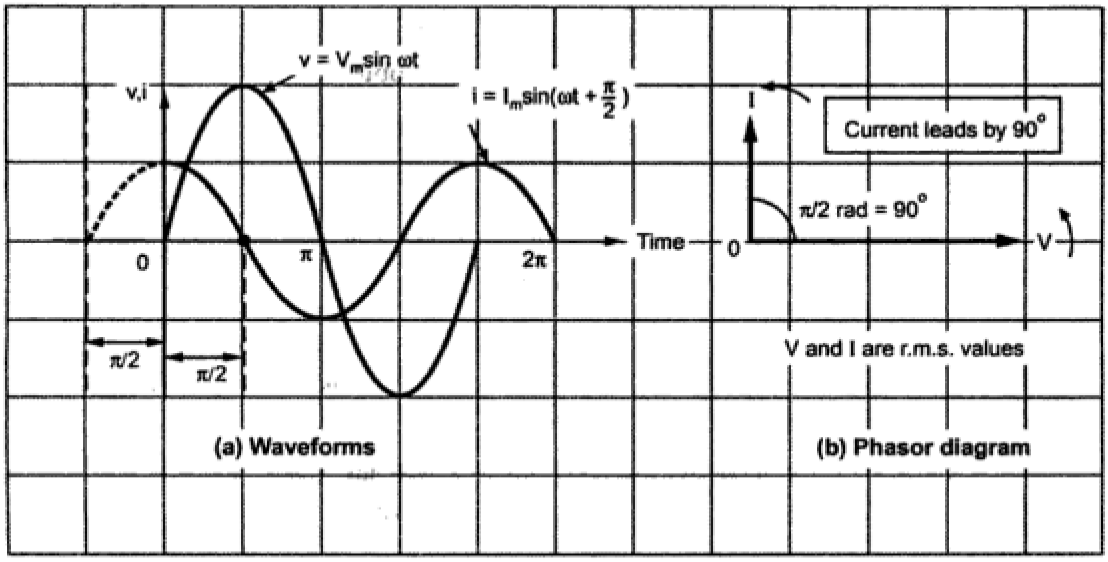
where $\quad \boxed{I_m = \frac{V_m}{X_C}}$

where $\quad \boxed{X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} \quad \Omega}$

The above equation clearly shows that the current is purely sinusoidal and having phase angle of $+\frac{\pi}{2}$ radians i.e. + 90°.

This means **current leads voltage applied by 90°**. The positive sign indicates leading nature of the current. If current is assumed reference, we can say that voltage across capacitor lags the current passing through the capacitor by 90°.

The Fig. 4.9 shows waveforms of voltage and current and the corresponding phasor diagram. The current waveform starts earlier by 90° in comparison with voltage waveform. When voltage is zero, the current has positive maximum value.



In purely capacitive circuit, current leads voltage by 90°.

Concept of Capacitive Reactance

We have seen while expressing current equation in the standard form that,

$$I_m = \frac{V_m}{X_C}$$

and

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}\ \Omega$$

The term X_C is called **Capacitive Reactance** and is measured in ohms.

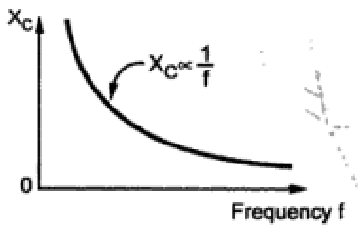


Fig. 4.10 X_C Vs f

So, **capacitive reactance** is defined as the opposition offered by the capacitance of a circuit to the flow of an alternating sinusoidal current.

X_C is measured in ohms and it depends on the frequency of the applied voltage.

The capacitive reactance is inversely proportional to the frequency for constant C.

$$X_C \propto \frac{1}{f} \qquad \text{for constant C}$$

The graph of X_C Vs f is a rectangular hyperbola as shown in Fig. 4.10.

Key Point : *If the frequency is zero, which is so for d.c. voltage, the capacitive reactance is infinite. Therefore, it is said that the capacitance offers open circuit to the d.c. or it blocks d.c.*

Power

The expression for the instantaneous power can be obtained by taking the product of instantaneous voltage and current.

$$P = v \times i = V_m \sin(\omega t) \times I_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$= V_m I_m \sin(\omega t) \cos(\omega t) \qquad \text{as } \sin\left(\omega t + \frac{\pi}{2}\right) = \cos \omega t$$

$$\therefore \quad \boxed{P = \frac{V_m I_m}{2} \sin(2\omega t)} \qquad \text{as } 2 \sin \omega t \cos \omega t = \sin 2\omega t$$

Thus, power curve is a sine wave of frequency double that of applied voltage. The average value of sine curve over a complete cycle is always zero.

$$P_{av} = \int_0^{2\pi} \frac{V_m I_m}{2} \sin(2\omega t)\, d(\omega t) = 0$$

The Fig. 4.11 shows waveforms of current, voltage and power. It can be observed from the figure that when power curve is positive, in practice, an electrostatic energy gets stored in the capacitor during its charging while the negative power curve represents that the energy stored is returned back to the supply during its discharging. The areas of positive and negative loops are exactly the same and hence, average power consumption is zero.

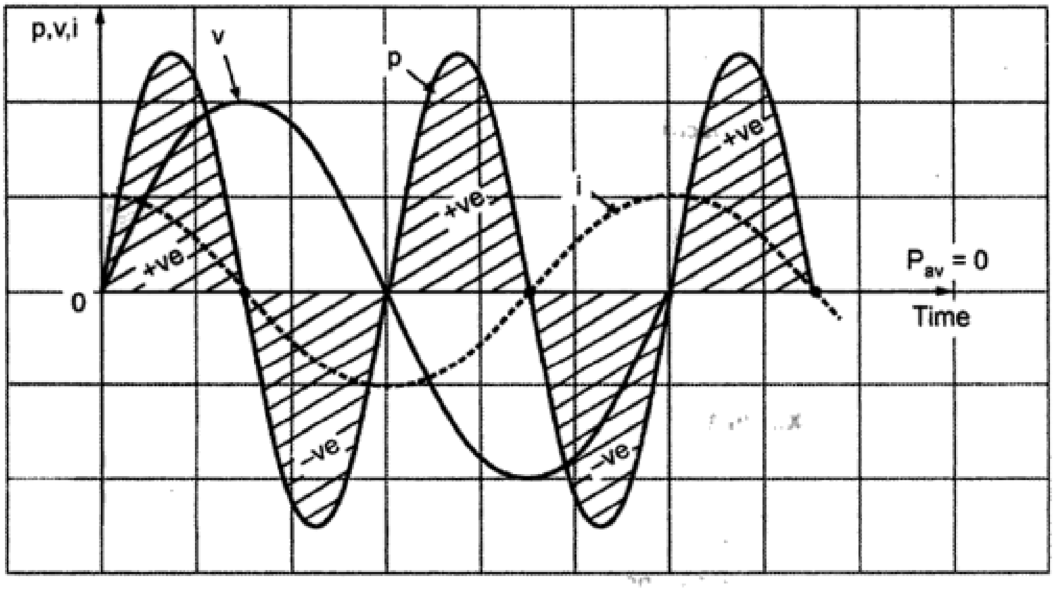


Fig. 4.11 Waveforms of voltage, current and power

Pure capacitance never consumes power.

Example 4.1 : *A 50 Hz, alternating voltage of 150 V (r.m.s.) is applied independently to (1) Resistance of 10 Ω (2) Inductance of 0.2 H (3) Capacitance of 50 μF*

Find the expression for the instantaneous current in each case. Draw the phasor diagram in each case.

Solution : Case 1 : R = 10 Ω

$$v = V_m \sin \omega t$$

$$V_m = \sqrt{2}\, V_{rms}$$

$$= \sqrt{2} \times 150 = 212.13 \text{ V}$$

$$I_m = \frac{V_m}{R} = \frac{212.13}{10}$$

$$= 21.213 \text{ A}$$

In pure resistive circuit, current is in phase with the voltage.

$$\therefore \quad \phi = \text{Phase Difference} = 0°$$

$$\therefore \quad i = I_m \sin \omega t$$

$$= I_m \sin (2 \pi f t)$$

$$\therefore \quad \mathbf{i = 21.213 \sin (100 \pi t) \text{ A}}$$

Fig. 4.12 (a)

The phasor diagram is shown in the Fig. 4.12 (a).

Case 2 : $L = 0.2\ \Omega$

Inductive reactance, $X_L = \omega L = 2\pi f L$

$$\therefore \quad X_L = 2\pi \times 50 \times 0.2 = 62.83\ \Omega$$

$$\therefore \quad I_m = \frac{V_m}{X_L} = \frac{212.13}{62.83} = 3.37\ \text{A}$$

In pure inductive circuit, current lags voltage by 90°.

$$\therefore \quad \phi = \text{Phase difference} = -90^\circ = \frac{\pi}{2}\ \text{rad}$$

$$\therefore \quad i = I_m \sin(\omega t - \phi)$$

$$\therefore \quad i = 3.37 \sin\left(100\pi t - \frac{\pi}{2}\right)\ \text{A}$$

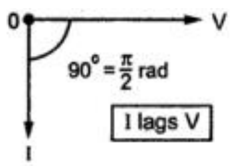


Fig. 4.12 (b)

The phasor diagram is shown in the Fig. 4.12 (b).

Case 3 : $C = 50\ \mu F$

Capacitive reactance, $X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$

$$\therefore \quad X_C = \frac{1}{2\pi \times 50 \times 50 \times 10^{-6}} = 63.66\ \Omega$$

$$\therefore \quad I_m = \frac{V_m}{X_C} = \frac{212.13}{63.66} = 3.33\ \text{A}$$

In pure capacitive circuit, current leads voltage by 90°.

$$\therefore \quad \phi = \text{Phase Difference} = 90^\circ = \frac{\pi}{2}\ \text{rad}$$

$$\therefore \quad i = I_m \sin(\omega t + \phi)$$

$$\therefore \quad i = 3.33 \sin\left(100\pi t + \frac{\pi}{2}\right)\ \text{A}$$

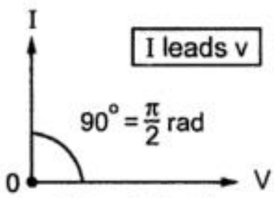


Fig. 4.12 (c)

The phasor diagram is shown in the Fig. 4.12 (c).

All the phasor diagrams represent r.m.s. values of voltage and current.

Example 4.2 : *A voltage $v = 141 \sin\{314t + \pi/3\}$ is applied to i) resistor of 20 ohms ii) inductance of 0.1 Henry iii) capacitance of 100 μF*

Find in each case r.m.s. value of current and power dissipated.

Draw the phasor diagram in each case.

Solution : Comparing given voltage with $v = V_m \sin(\omega t+\theta)$ we get,

$$V_m = 141 \text{ V and hence } V = V_{rms} = \frac{V_m}{\sqrt{2}} = 99.702 \text{ V}$$

$$\omega = 314 \text{ and hence } f = \frac{\omega}{2\pi} = 50 \text{ Hz}, \theta = \frac{\pi}{3} = 60°$$

Hence the polar form of applied voltage becomes,

$$V = 99.702 \angle 60° \text{ V}$$

Case 1 : $R = 20\ \Omega$

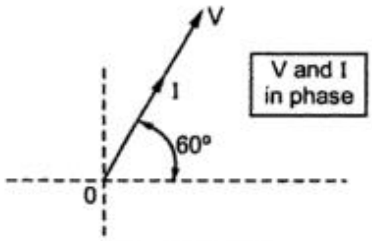


Fig. 4.13 (a)

$$I = \frac{V}{R} = \frac{99.702\angle 60°}{20\angle 0°} = 4.9851 \angle 60° \text{ A}$$

$\therefore$ I_{rms} = **4.9851 A**

The phase of both V and I is same for pure resistive circuit. Both are in phase.

$P = VI = 99.702 \times 4.9851 =$ **497.0244 W**

The phasor diagram is shown in the Fig. 4.13 (a).

Case 2 : $L = 0.1$ H

$\therefore$ $X_L = \omega L = 314 \times 0.1 = 31.4\ \Omega$

$\therefore$ $I = \frac{|V|}{X_L} = \frac{99.702}{31.4} =$ **3.1752 A**

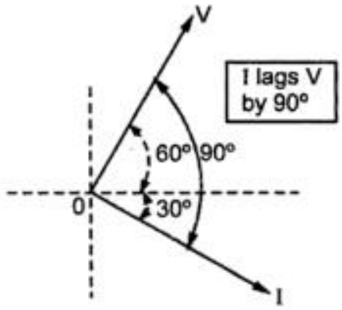


Fig. 4.13 (b)

This is r.m.s. value of current. It has to lag the applied voltage by 90° in case of pure inductor.

Hence phasor diagram is shown in the Fig. 4.13 (b).

The individual phase of I is − 30°.

In polar form I can be represented as $3.1752\angle -30°$ A.

Pure inductor never consumes power so **power dissipated is zero.**

Case 3 : $C = 100$ μF

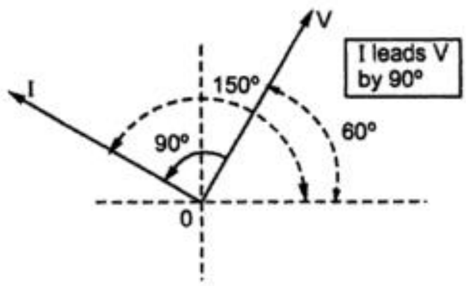


Fig. 4.13 (c)

$$\therefore\ X_C = \frac{1}{\omega C} = \frac{1}{314 \times 100 \times 10^{-6}} = 31.8471\ \Omega$$

$$\therefore\quad I = \frac{|V|}{X_C} = \frac{99.702}{31.8471} = \mathbf{3.1306\ A}$$

This is r.m.s. value of current.

It has to lead the applied voltage by 90° in case of pure capacitor.

Hence phasor diagram is shown in the Fig. 4.13 (c).

The individual phase of I is 150°. In polar form I can be represented as $3.1306\angle +150°$ A. Pure capacitor never consumes power and hence **power dissipated is zero.**

A.C. Through Series R-L Circuit

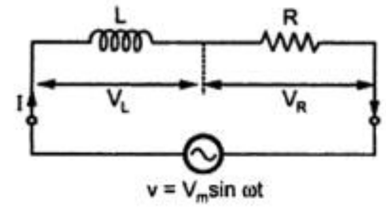


Fig. 4.14 (a) Series R-L circuit

Consider a circuit consisting of pure resistance R ohms connected in series with a pure inductance of L henries as shown in the Fig. 4.14 (a).

The series combination is connected across a.c. supply given by $v = V_m \sin \omega t$.

Circuit draws a current I then there are two voltage drops,

a) Drop across pure resistance, $V_R = I \times R$

b) Drop across pure inductance, $V_L = I \times X_L$ where $X_L = 2 \pi f L$

I = r.m.s. value of current drawn

V_R , V_L = r.m.s. values of the voltage drops.

The Kirchhoff's voltage law can be applied to the a.c. circuit but only the point to remember is the addition of voltages should be a phasor (vector) addition and no longer algebraic as in case of d.c.

$$\therefore \quad \overline{V} = \overline{V_R} + \overline{V_L} \qquad \text{(phasor addition)}$$

$$\therefore \quad \overline{V} = \overline{IR} + \overline{I X_L}$$

Let us draw the phasor diagram for the above case.

Key Point : *For series a.c. circuits, generally, current is taken as the reference phasor as it is common to both the elements.*

Following are the **steps to draw the phasor diagram :**

1) Take current as a reference phasor.

2) In case of resistance, voltage and current are in phase, so V_R will be along current phasor.

3) In case of inductance, current lags voltage by 90°. But, as current is reference, V_L must be shown leading with respect to current by 90°.

4) The supply voltage being vector sum of these two vectors V_L and V_R obtained by law of parallelogram.

From the voltage triangle, we can write,

$$V = \sqrt{(V_R)^2 + (V_L)^2} = \sqrt{(IR)^2 + (I \times X_L)^2}$$

$$= I \sqrt{(R)^2 + (X_L)^2}$$

$$\therefore \quad V = I Z$$

where $\boxed{Z = \sqrt{(R)^2 + (X_L)^2}}$ ... **Impedance** of the circuit.

The impedance Z is measured in ohms.

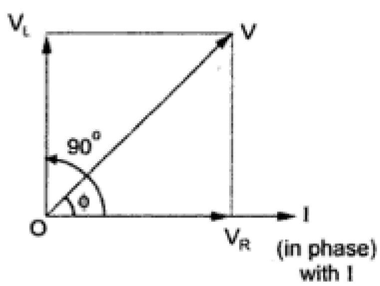


Fig. 4.14 (b) Phasor diagram

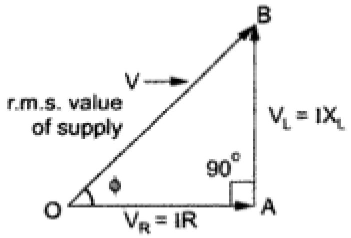


Fig. 4.14 (c) Voltage triangle

Impedance

Impedance is defined as the opposition of circuit to flow of alternating current. It is denoted by Z and its unit is ohms.

For the R-L series circuit, it can be observed from the phasor diagram that the current lags behind the applied voltage by an angle ϕ. From the voltage triangle, we can write,

$$\tan\phi = \frac{V_L}{V_R} = \frac{X_L}{R}, \quad \cos\phi = \frac{V_R}{V} = \frac{R}{Z}, \quad \sin\phi = \frac{V_L}{V} = \frac{X_L}{Z}$$

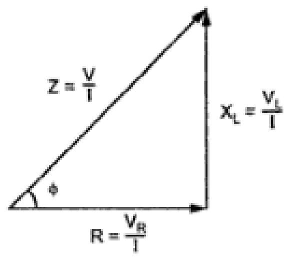


Fig. 4.15 Impedance triangle

If all the sides of the voltage triangle are divided by current, we get a triangle called **impedance triangle** as shown in the Fig. 4.15.

Sides of this triangle are resistance R, inductive reactance X_L and an impedance Z.

From this impedance triangle, we can see that the X component of impedance is R and is given by,

$$R = Z\cos\phi$$

and Y component of impedance is X_L and is given by,

$$X_L = Z\sin\phi$$

In rectangular form the impedance is denoted as,

$$\boxed{Z = R + j X_L \quad \Omega}$$

While in polar form, it is denoted as,

$$Z = |Z| \angle\phi \quad \Omega$$

where

$$|Z| = \sqrt{R^2 + X_L^2}, \qquad \phi = \tan^{-1}\left[\frac{X_L}{R}\right]$$

Key Point: *Thus ϕ is + ve for inductive impedance.*

Power and Power Triangle

The expression for the current in the series R-L circuit is,

$$i = I_m \sin(\omega t - \phi) \text{ as current lags voltage.}$$

The power is product of instantaneous values of voltage and current,

$$\therefore \quad P = v \times i = V_m \sin\omega t \times I_m \sin(\omega t - \phi)$$

$$= V_m I_m [\sin(\omega t) \,.\, \sin(\omega t - \phi)]$$

$$= V_m I_m \left[\frac{\cos(\phi) - \cos(2\omega t - \phi)}{2}\right]$$

$$= \frac{V_m I_m}{2}\cos\phi - \frac{V_m I_m}{2}\cos(2\omega t - \phi)$$

Now, the second term is cosine term whose average value over a cycle is zero. Hence, average power consumed is,

$$P_{av} = \frac{V_m I_m}{2}\cos\phi = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}}\cos\phi$$

$$\therefore \quad \boxed{P = V I \cos\phi \quad \text{watts}}$$ where V and I are r.m.s. values

If we multiply voltage equation by current I, we get the power equation.

$$\overline{VI} = \overline{V_R I} + \overline{V_L I}$$

$$\therefore \quad \overline{VI} = \overline{V\cos\phi I} + \overline{V\sin\phi I}$$

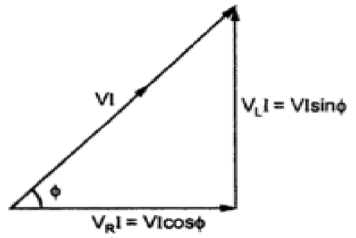


Fig. 4.16 Power triangle

From this equation, power triangle can be obtained as shown in the Fig. 4.16.

So, three sides of this triangle are,

1) VI, 2) VI cos ϕ, 3) VI sin ϕ

These three terms can be defined as below.

Apparent Power (S)

It is defined as the product of r.m.s. value of voltage (V) and current (I). It is denoted by S.

$$\therefore \quad \boxed{S = VI \quad \text{VA}}$$

It is measured in unit volt-amp (VA) or kilo volt-amp (kVA).

Real or True Power (P)

It is defined as the product of the applied voltage and the active component of the current.

It is real component of the apparent power. It is measured in unit watts (W) or kilowatts (kW).

$$\boxed{P = VI\cos\phi \quad \text{watts}}$$

Reactive Power (Q)

It is defined as product of the applied voltage and the reactive component of the current.

It is also defined as imaginary component of the apparent power. It is represented by 'Q' and it is measured in unit Volt-Amp Reactive (VAR) or kilo volt–Amp Reactive kVAR

$$\boxed{Q = VI\sin\phi \quad \text{VAR}}$$

Apparent power, $S = VI$ VA

True power $P = VI\cos\phi$ W (Average Power)

Reactive power $Q = VI\sin\phi$ VAR

Power Factor (cos ϕ)

It is defined as factor by which the apparent power must be multiplied in order to obtain the true power.

It is the ratio of true power to apparent power.

$$\boxed{\text{Power factor} = \frac{\text{True Power}}{\text{Apparent Power}} = \frac{VI\cos\phi}{VI} = \cos\phi}$$

The numerical value of cosine of the phase angle between the applied voltage and the current drawn from the supply voltage gives the power factor. It cannot be greater than 1.

It is also defined as the ratio of resistance to the impedance.

$$\boxed{\cos\phi = \frac{R}{Z}}$$

Key Point : *The nature of power factor is always determined by position of current with respect to the voltage.*

If current lags voltage power factor is said to be lagging. If current leads voltage power factor is said to be leading.

So, for pure inductance, the power factor is cos (90°) i.e. zero lagging while for pure capacitance, the power factor is cos (90°) i.e. zero but leading. For purely resistive circuit voltage and current are in phase i.e. $\phi = 0$. Therefore, power factor is cos (0°) = 1. Such circuit is called unity power factor circuit.

$$\boxed{\text{Power factor} = \cos\phi}$$

ϕ is the angle between supply voltage and current.

Key Point : *Nature of power factor always tells position of current with respect to voltage.*

Example 4.3 : *An alternating current, $i = 1.414 \sin (2\pi \times 50 \times t)$ A, is passed through a series circuit consisting of a resistance of 100 ohm and an inductance of 0.31831 henry. Find the expressions for the instantaneous values of the voltages across (i) the resistance, (ii) the inductance and (iii) the combination.*

Solution : The circuit is shown in the Fig. 4.17.

$$i = 1.414 \sin (2\pi \times 50\, t) \text{ A}$$

$$\therefore \quad \omega = 2\pi \times 50 = 2\pi f$$

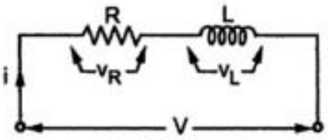


Fig. 4.17

$$\therefore \quad f = 50 \text{ Hz}, \quad R = 100\ \Omega, \quad L = 0.31831 \text{ H}$$

$$\therefore \quad X_L = 2\pi f L$$

$$= 2\pi \times 50 \times 0.31831 = 100\ \Omega$$

i) The voltage across the resistance is,

$$v_R = i R = 1.414 \sin (2\pi \times 50\, t) \times 100$$

$$= \mathbf{141.4 \sin (2\pi \times 50\, t)\ V}$$

ii) The voltage across L leads current by 90° as current lags by 90° with respect to voltage.

$$\therefore \quad v_L = i X_L \text{ but leading current by } 90°$$

$$= \mathbf{141.4 \sin (2\pi \times 50\, t + 90°)}\ \text{V}$$

iii) From the expression of V_R we can write,

$$\text{r.m.s. value of } V_R = \frac{141.4}{\sqrt{2}} = 100 \text{ V}, \qquad \phi = 0°$$

$$\therefore \quad V_R = 100 \angle 0° = 100 + j0 \text{ V}$$

$$\text{r.m.s. value of } V_L = \frac{141.4}{\sqrt{2}} = 100 \text{ V}, \qquad \phi = 90°$$

$$\therefore \quad V_L = 100 \angle 90° = 0 + j\,100 \text{ V}$$

$$\therefore \quad V = \bar{V}_R + \bar{V}_L = 100 + j0 + 0 + j100$$

$$= 100 + j\,100 = 141.42 \angle 45° \text{ V}$$

$$\therefore \quad V_m = \sqrt{2} \times 141.42 = 200 \text{ V}$$

Hence expression of instantaneous value of resultant voltage is,

$$v = \mathbf{200 \sin (2\pi \times 50\, t + 45°)\ V}$$

Example. 4.4 : *A voltage $e = 200 \sin 100\pi t$ is applied to a load having $R = 200\ \Omega$ in series with $L = 638$ mH.*

Estimate :-

i) expression for current in $i = I_m \sin (\omega t \pm \phi)$ form ii) power consumed by the load iii) reactive power of the load iv) voltage across R and L.

Solution : The circuit is shown in the Fig. 4.18.

$$e_m = 200 \text{ V} \quad \therefore \ V = \frac{200}{\sqrt{2}} = 141.421 \text{ V (r.m.s.)}$$

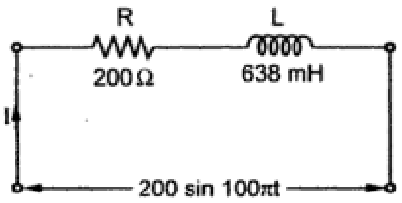


Fig. 4.18

$$\omega = 100\pi \quad \therefore \ f = 50 \text{ Hz}$$

$$\therefore \quad X_L = \omega L = 100\pi \times 638 \times 10^{-3}$$

$$= 200.433\ \Omega$$

$$Z = R + j X_L = 200 + j\,200.433\ \Omega$$

$$= 283.149 \angle 45.06°\ \Omega$$

$\therefore \qquad I = \dfrac{V}{Z} = \dfrac{141.421\angle 0^\circ}{283.149\angle 45.06^\circ}$

$= 0.5\angle -45.06^\circ$ A, current lags voltage by 45.06°.

$\therefore \qquad I_m = \sqrt{2}\times 0.5 = 0.7071$ A, $\phi = -45.06^\circ$

i) $\quad i = I_m \sin(\omega t - \phi)$

$= \mathbf{0.7071 \sin(100\,\pi\, t - 45.06^\circ)\ A}$

ii) $\quad P = VI\cos\phi = 141.421\times 0.5\times\cos(45.06^\circ)$

$= 49.9474 \approx \mathbf{50\ W}$

iii) $\quad Q = VI\sin\phi = 141.421\times 0.5\times\sin(45.06^\circ)$

$= \mathbf{50\ VAR}$

iv) $\quad V_R = IR = 0.5\times 200 = \mathbf{100\ V}$

$V_L = I X_L = 0.5\times 200.433 = \mathbf{100.21\ V}$

A.C. Through Series R-C Circuit

Consider a circuit consisting of pure resistance R-ohms and connected in series with a pure capacitor of C-farads as shown in the Fig. 4.19.

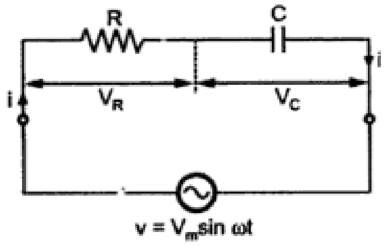


Fig. 4.19 Series R-C circuit

The series combination is connected across a.c. supply given by

$$v = V_m \sin \omega t$$

Circuit draws a current I, then there are two voltage drops,

a) Drop across pure resistance $\quad V_R = I \times R$

b) Drop across pure capacitance $\quad V_C = I \times X_C$

where $\quad X_C = \dfrac{1}{2\pi f C}$ and I, V_R, V_C are the r.m.s. values

The Kirchhoff's voltage law can be applied to get,

$$V = \overline{V_R} + \overline{V_C} \qquad \text{... (Phasor Addition)}$$

$$\therefore \qquad \overline{V} = \overline{IR} + \overline{IX_C}$$

Let us draw the phasor diagram. Current I is taken as reference as it is common to both the elements.

Following are the **steps to draw the phasor diagram** :-

1) Take current as reference phasor.
2) In case of resistance, voltage and current are in phase. So, V_R will be along current phasor.
3) In case of pure capacitance, current leads voltage by 90° i.e. voltage lags current by 90° so V_C is shown downwards i.e. lagging current by 90°.
4) The supply voltage being vector sum of these two voltages V_C and V_R obtained by completing parallelogram.

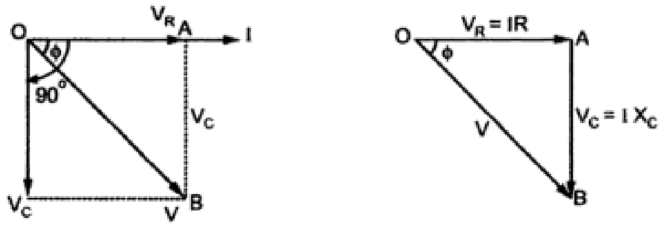


Fig. 4.20 Phasor diagram and voltage triangle

From the voltage triangles,

$$V = \sqrt{(V_R)^2 + (V_C)^2} = \sqrt{(IR)^2 + (I X_C)^2}$$

$$= I\sqrt{(R)^2 + (X_C)^2}$$

$$\therefore \quad V = I Z$$

where $\boxed{Z = \sqrt{(R)^2 + (X_C)^2}}$ is the impedance of the circuit.

Impedance

It is measured in ohms given by $Z = \sqrt{(R)^2 + (X_C)^2}$ where $X_C = \frac{1}{2\pi f C}$ Ω called **capacitive reactance.**

In R-C series circuit, current leads voltage by angle ϕ or supply voltage V lags current I by angle ϕ as shown in the phasor diagram in Fig. 4.21.

From voltage triangle, we can write,

$$\tan\phi = \frac{V_C}{V_R} = \frac{X_C}{R}, \cos\phi = \frac{V_R}{V} = \frac{R}{Z}, \quad \sin\phi = \frac{V_C}{V} = \frac{X_C}{Z}$$

If all the sides of the voltage triangle are divided by the current, we get a triangle called **impedance triangle.**

Two sides of the triangle are 'R' and 'X_C' and the third side is impedance 'Z'.

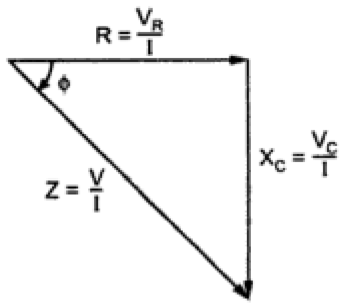


Fig. 4.21 Impedance triangle

The X component of impedance is R and is given by

$$R = Z\cos\phi$$

and Y component of impedance is X_C and is given by

$$X_C = Z\sin\phi$$

But, as direction of the X_C is the negative Y direction, the rectangular form of the impedance is denoted as,

$$Z = R - j X_C \ \Omega$$

While in polar form, it is denoted as,

where $\boxed{Z = |Z| \angle -\phi \ \Omega}$

$$Z = R - jX_C = |Z| \angle -\phi$$

where $|Z| = \sqrt{R^2 + X_C^2}, \ \phi = \tan^{-1}\left[\frac{-X_C}{R}\right]$

Key Point : *Thus ϕ is – ve for capacitive impedance.*

Power and Power Triangle

The current leads voltage by angle ϕ, hence its expression is,

$$i = I_m \sin(\omega t + \phi) \text{ as current leads voltage}$$

The power is the product of instantaneous values of voltage and current.

$\therefore \quad P = v \times i = V_m \sin \omega t \times I_m \sin(\omega t + \phi)$

$$= V_m I_m [\sin(\omega t) \cdot \sin(\omega t + \phi)]$$

$$= V_m I_m \left[\frac{\cos(-\phi) - \cos(2\omega t + \phi)}{2}\right]$$

$$= \frac{V_m I_m \cos\phi}{2} - \frac{V_m I_m}{2} \cos(2\omega t + \phi) \qquad \text{as } \cos(-\phi) = \cos\phi$$

Now, second term is cosine term whose average value over a cycle is zero. Hence, average power consumed by the circuit is,

$$P_{av} = \frac{V_m I_m}{2} \cos\phi = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos\phi$$

$\therefore \quad \boxed{P = V I \cos\phi \quad \text{watts}}$ where V and I are r.m.s. values

If we multiply voltage equation by current I, we get the power equation,

$$\overline{VI} = \overline{V_R I} + \overline{V_C I}$$

$$\therefore \quad \overline{VI} = \overline{VI\cos\phi} + \overline{VI\sin\phi}$$

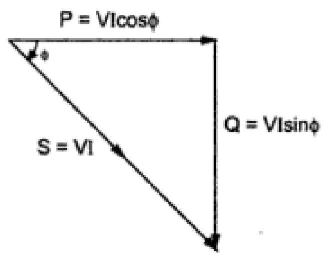


Fig. 4.22

Hence, the power triangle can be shown as in the Fig. 4.22.

Thus, the various powers are,

Apparent Power, $S = V I$ V A

True or average power, $P = V I \cos\phi$ W

Reactive Power, $Q = V I \sin\phi$ V A R

Remember that, X_L term appears positive in Z.

$$Z = R + j X_L = |Z| \angle \phi \qquad \phi \text{ is + ve for inductive Z}$$

While X_C term appears negative in Z.

$$Z = R - j X_C = |Z| \angle -\phi \qquad \phi \text{ is – ve for capacitive Z}$$

For any single phase a.c. circuit, the average power is given by,

$$\mathbf{P = V I \cos\phi \quad watts}$$

where V, I are r.m.s. values

$$\cos\phi = \text{Power factor of circuit}$$

$\cos\phi$ **is lagging for inductive** circuit and $\cos\phi$ **is leading for capacitive** circuit.

Example 4.5 : *Calculate the resistance and inductance or capacitance in series for each of the following impedances. Assume the frequency to be 60 Hz.*

i) (12 + j 30) ohms ii) – j 60 ohms iii) 20 ∠ 60° ohms.

Solution : i) $12 + j\,30\ \Omega$

Comparing the value of impedance with,

$$Z = R + j X_L, \quad R = 12\ \Omega \quad \text{and } X_L = 30\ \Omega = 2\pi f L$$

$$\therefore \quad L = \frac{30}{2\pi f} = \frac{30}{2\pi \times 60} = \mathbf{79.58\ mH}$$

ii) $0 - j\,60\ \Omega$

Comparing with, $Z = R - j X_C$

$$R = \mathbf{0\ \Omega}$$

$$X_C = 60\ \Omega = \frac{1}{2\pi f C}$$

$$\therefore \quad C = \frac{1}{2\pi \times 60 \times 60} = \mathbf{44.209\ \mu F}$$

iii) $20 \angle 60°\ \Omega$

Converting to rectangular form, $Z = 10 + j\,17.32$

Comparing with, $Z = R + j X_L$

$$R = \mathbf{10\ \Omega}$$

$$X_L = 17.32\ \Omega = 2\pi f L$$

$$\therefore \quad L = \frac{1732}{2\pi \times 60} = \mathbf{45.94\ mH}$$

Example 4.6 : *A resistance of 120 ohms and a capacitive reactance of 250 ohms are connected in series across a AC voltage source. If a current of 0.9 A is flowing in the circuit find out (i) power factor, (ii) supply voltage (iii) voltages across resistance and capacitance (iv) Active power and reactive power.*

Solution : The circuit is shown in the Fig. 4.23.

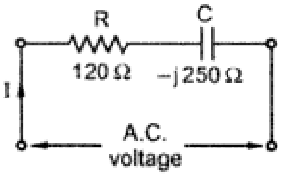


Fig. 4.23

$R = 120\ \Omega,\ X_C = 250\ \Omega,\ I = 0.9\ A$

$Z = R - j\,X_C = 120 - j250\ \Omega = 277.308\ \angle -64.358°$

Take current as reference.

$\therefore\ I = 0.9\ \angle\ 0°\ A$

i) Power factor $\cos\phi = \cos(-64.358°) = \mathbf{0.4327}$ **leading**

ii) Supply voltage $V = I \times Z = [0.9\ \angle 0°] \times [277.308\ \angle -64.358°]$

$\therefore\ V = \mathbf{249.5772\ \angle -64.358°\ V}$

iii) $V_R = I \times R = 0.9 \times 120 = \mathbf{108\ V}$ (magnitude)

$V_C = I \times X_C = 0.9 \times 250 = \mathbf{225\ V}$ (magnitude)

iv) P = Active power = $V I \cos\phi = 249.5772 \times 0.9 \times 0.4327$

$= \mathbf{97.1928\ W}$

Q = Reactive power = $VI \sin\phi$

$= 249.5772 \times 0.9 \times \sin(-64.358°)$

$= \mathbf{-202.498\ VAR}$

The negative sign indicates leading nature of reactive volt-amperes.

A.C. Through Series R-L-C Circuit

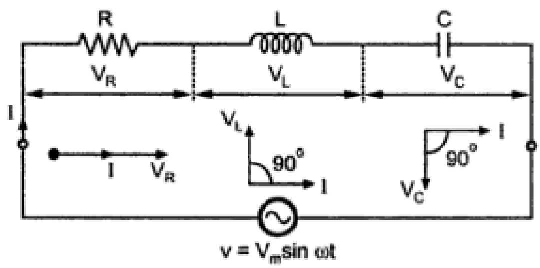


Fig. 4.24 R-L-C series circuit

Consider a circuit consisting of resistance R ohms pure inductance L henries and capacitance C farads connected in series with each other across a.c. supply. The circuit is shown in the Fig. 4.24.

The a.c. supply is given by,

$v = V_m \sin \omega t$. The circuit draws a current I. Due to current I, there are different voltage drops across R, L and C which are given by,

a) Drop across resistance R is $V_R = I\,R$

b) Drop across inductance L is $V_L = I\,X_L$

c) Drop across capacitance C is $V_C = I\,X_C$

The values of I, V_R, V_L and V_C are r.m.s. values

The characteristics of three drops are,

a) V_R is in phase with current I.

b) V_L leads current I by 90°.

c) V_C lags current I by 90°.

According to Kirchhoff's laws, we can write,

$$\bar{V} = \bar{V}_R + \bar{V}_L + \bar{V}_C \quad \ldots \text{Phasor addition}$$

Let us see the phasor diagram. Current I is taken as reference as it is common to all the elements.

Following are the steps to draw the phasor diagram :

1) Take current as reference. 2) V_R is in phase with I.

3) V_L leads current I by 90°. 4) V_C lags current I by 90°.

5) Obtain the resultant of V_L and V_C. Both V_L and V_C are in phase opposition (180° out of phase).

6) Add that with V_R by law of parallelogram to get the supply voltage.

The phasor diagram depends on the conditions of the magnitudes of V_L and V_C which ultimately depends on the values of X_L and X_C. Let us consider the different cases.

1. **$X_L > X_C$**

When $X_L > X_C$, obviously, I X_L i.e. V_L is greater than I X_C i.e. V_C. So, resultant of V_L and V_C will be directed towards V_L i.e. leading current I. Current I will lag the resultant of V_L and V_C i.e. $(V_L - V_C)$.

The circuit is said to be **inductive** in nature. The phasor sum of V_R and $(V_L - V_C)$ gives the resultant supply voltage, V. This is shown in the Fig. 4.25.

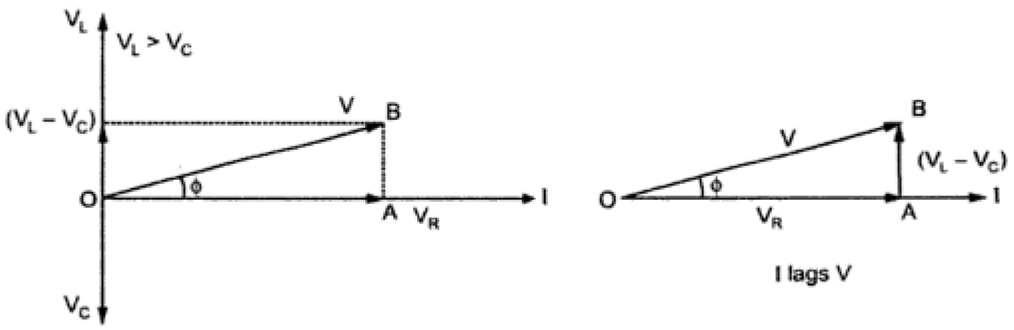


Fig. 4.25 Phasor diagram and voltage triangle for $X_L > X_C$

From the voltage triangle, $V = \sqrt{(V_R)^2 + (V_L - V_C)^2} = \sqrt{(I\,R)^2 + (I\,X_L - I\,X_C)^2}$

$$= I\sqrt{(R)^2 + (X_L - X_C)^2}$$

$\therefore$ $$V = I\,Z$$

where $$Z = \sqrt{(R)^2 + (X_L - X_C)^2}$$

So, if $v = V_m \sin \omega t$, then $i = I_m \sin(\omega t - \phi)$ as current lags voltage by angle ϕ.

2. **$X_L < X_C$**

When $X_L < X_C$, obviously, I X_L i.e. V_L is less than I X_C i.e. V_C. So, the resultant of V_L and V_C will be directed towards V_C. Current I will lead $(V_C - V_L)$.

The circuit is said to be **capacitive** in nature. The phasor sum of V_R and $(V_C - V_L)$ gives the resultant supply voltage V. This is shown in the Fig. 4.26.

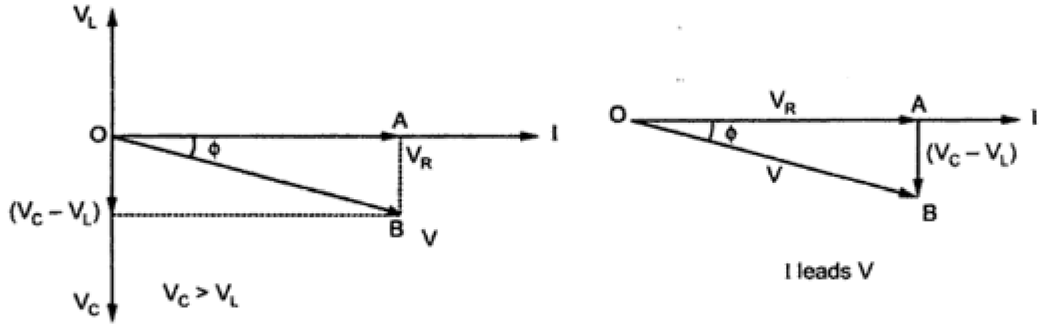


Fig. 4.26 Phasor diagram and voltage triangle for $X_L < X_C$

From the voltage triangle,

$$V = \sqrt{(V_R)^2 + (V_C - V_L)^2} = \sqrt{(I\,R)^2 + (I\,X_C - I\,X_L)^2}$$

$$= I\sqrt{(R)^2 + (X_C - X_L)^2}$$

$\therefore$ $$V = I\,Z$$

where $$Z = \sqrt{(R)^2 + (X_C - X_L)^2}$$

So, if $v = V_m \sin \omega t$, then $i = I_m \sin(\omega t + \phi)$ as current leads voltage by angle ϕ.

3. $X_L = X_C$

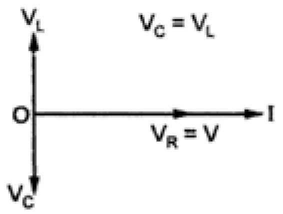


Fig. 4.27 Phasor diagram for $X_L = X_C$

When $X_L = X_C$, obviously, $V_L = V_C$. So, V_L and V_C will cancel each other and their resultant is zero. So, $V_R = V$ in such case and overall circuit is purely resistive in nature. The phasor diagram is shown in the Fig. 4.27.

From phasor diagram, $V = V_R$

$\therefore$ $$V = I\,R$$

$\therefore$ $$V = I\,Z$$

where $$Z = R$$

Impedance

In general, for RLC series circuit impedance is given by,

$$Z = R + j\,X$$

where $X = X_L - X_C$ = total reactance of circuit

If $X_L > X_C$, X is positive and circuit is inductive.

If $X_L < X_C$, X is negative and circuit is capacitive.

If $X_L = X_C$, X is zero and circuit is purely resistive.

$$\tan\phi = \left[\frac{X_L - X_C}{R}\right], \quad \cos\phi = \frac{R}{Z} \text{ and } Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Impedance Triangle

The impedance is expressed as,

$$Z = R + jX \quad \text{where} \quad X = X_L - X_C$$

For $X_L > X_C$, ϕ is positive and the impedance triangle is as shown in the Fig. 4.28 (a).

For $X_L < X_C$, $X_L - X_C$ is negative, so ϕ is negative and the impedance triangle is as shown in Fig. 4.28 (b).

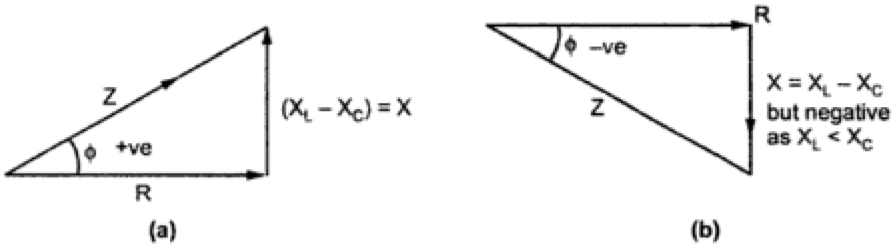


Fig. 4.28 Impedance triangles

In both the cases, $R = Z\cos\phi$ and $X = Z\sin\phi$

Power and Power Triangle

The average power consumed by the circuit is,

$$P_{av} = \text{Average power consumed by R+ Average power consumed by L}$$
$$+ \text{ Average power consumed by C}$$

But, pure L and C never consume any power.

$$\therefore \quad P_{av} = \text{Power taken by R} = I^2 R = I\,(I\,R) = I\,V_R$$

$$\text{But,} \quad V_R = V\cos\phi \text{ in both the cases}$$

$$\therefore \quad P = V\,I\cos\phi \text{ W}$$

Thus, for any condition, $X_L > X_C$ or $X_L < X_C$, the power can be expressed as,

$$\boxed{P = \text{Voltage} \times \text{Component of current in phase with voltage}}$$

Key Point : *The power triangle can be obtained by multiplying each side of impedance triangle by I^2.*

The power triangles are shown in the Fig. 4.29.

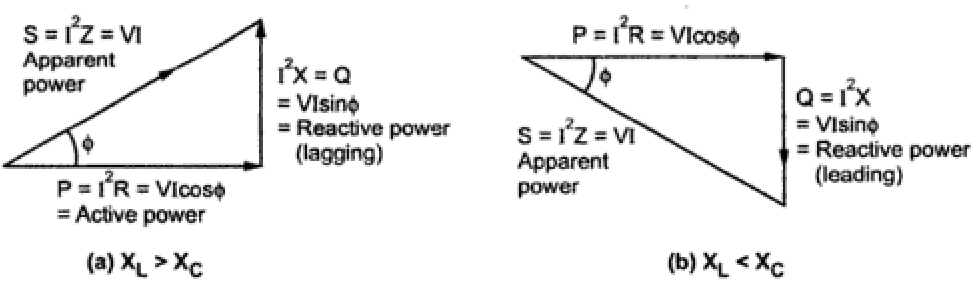


Fig. 4.29

Summary of R, L and C circuits

No.	Circuit	Impedance (Z)		ϕ	p.f. $\cos\phi$	Remark
		Polar	Rectangular			
1.	Pure R	$R\angle 0°\ \Omega$	$R + j0\ \Omega$	0°	1	Unity p.f.
2.	Pure L	$X_L\angle 90°\ \Omega$	$0 + j X_L\ \Omega$	90°	0	Zero lagging
3.	Pure C	$X_C\angle -90°\ \Omega$	$0 - j X_C\ \Omega$	– 90°	0	Zero leading
4.	Series RL	$\lvert Z\rvert\angle +\phi°\ \Omega$	$R + j X_L\ \Omega$	$0° \angle \phi \angle 90°$	$\cos\phi$	Lagging
5.	Series RC	$\lvert Z\rvert\angle -\phi°\ \Omega$	$R - j X_C\ \Omega$	$-90° \angle \phi \angle 0°$	$\cos\phi$	Leading
6.	Series RLC	$\lvert Z\rvert\angle \pm\phi°\ \Omega$	$R + j X\ \Omega$ $X = X_L - X_C$	ϕ	$\cos\phi$	$X_L > X_C$ Lagging $X_L < X_C$ Leading $X_L = X_C$ Unity

Example 4.7 : *A series circuit consisting of 25 Ω resistor, 64 mH inductor and 80 μF capacitor, is connected to a 110 V, 50 Hz, single phase supply as shown in Fig. 4.30. Calculate the current, voltage across individual element and the overall p.f. of the circuit. Draw a neat phasor diagram showing $\bar{I}$, $\bar{V}_R$, $\bar{V}_L$, $\bar{V}_C$ and $\bar{V}$.*

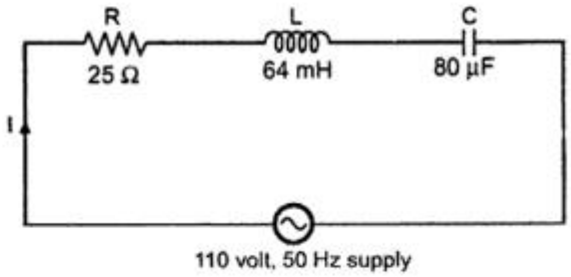


Fig. 4.30

Solution : From Fig. 4.30,

$$R = 25\ \Omega$$

$$X_L = 2\pi f L = 2\pi\times 50\times 64\times 10^{-3} = 20.10\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi\times 50\times 80\times 10^{-6}} = 39.78\ \Omega$$

$$Z = R + jX_L - jX_C = 25 + j\,20.10 - j\,39.78$$

$$Z = (25 - j19.68)\ \Omega$$

$$I = \frac{V}{Z} = \frac{110\angle 0°}{25 - j\,19.68} = \frac{110\angle 0°}{31.81\angle -38.20°}$$

$$I = \mathbf{3.4580\angle 38.20°\ A}$$

$$\therefore \quad I = 3.4580\ \text{A}$$

$$V_R = IR = (3.4580\angle 38.20°)(25) = 86.45\angle 38.20° \text{ volts}$$

$$V_L = I(jX_L) = (3.4580\angle 38.20°)(j20.10)$$

$$= (3.4580\angle 38.20°)(20.10\angle 90°) = \mathbf{69.50\angle 128.2°} \text{ volts}$$

$$V_C = I(-jX_C) = (3.4580\angle 38.20°)(-j\,39.78)$$

$$= (3.4580\angle 38.20°)(38.78\angle -90°) = \mathbf{134.10\angle -51.9°\ volts}$$

$$V = 110\angle 0° \text{ volts}$$

Overall p.f., $\cos\phi = \cos 38.20° = 0.7858$ **leading.**

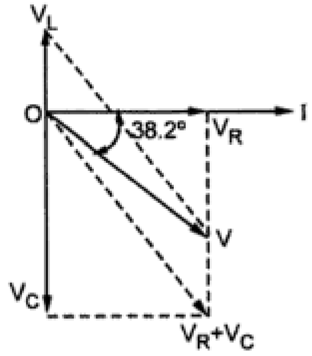


Fig. 4.30 (a)

A. C. Parallel Circuit

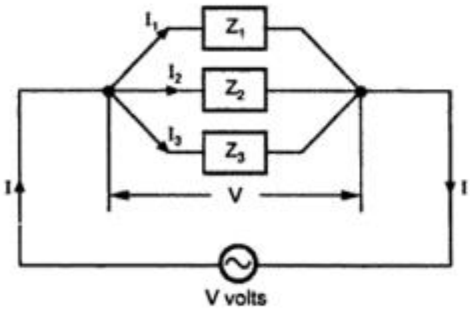


Fig. 4.33 A.C. parallel circuit

A parallel circuit is one in which two or more impedances are connected in parallel across the supply voltage. Each impedance may be a separate series circuit. Each impedance is called branch of the parallel circuit.

The Fig. 4.33 shows a parallel circuit consisting of three impedances connected in parallel across an a.c. supply of V volts.

Key Point : *The voltage across all the impedances is same as supply voltage of V volts.*

The current taken by each impedance is different.

Applying Kirchhoff's law, $\bar{I} = \bar{I_1} + \bar{I_2} + \bar{I_3}$ ... (phasor addition)

$$\therefore \quad \frac{\bar{V}}{\bar{Z}} = \frac{\bar{V}}{\bar{Z_1}} + \frac{\bar{V}}{\bar{Z_2}} + \frac{\bar{V}}{\bar{Z_3}}$$

$$\therefore \quad \frac{1}{\bar{Z}} = \frac{1}{\bar{Z_1}} + \frac{1}{\bar{Z_2}} + \frac{1}{\bar{Z_3}}$$

where Z is called **equivalent impedance.** This result is applicable for 'n' such impedances connected in parallel.

Following are the steps to solve parallel a.c. circuit :

1) The currents in the individual branches are to be calculated by using the relation

$$\bar{I_1} = \frac{\bar{V}}{\bar{Z_1}}, \quad \bar{I_2} = \frac{\bar{V}}{\bar{Z_2}}, \ldots\ldots, \quad \bar{I_n} = \frac{\bar{V}}{\bar{Z_n}}$$

while the individual phase angles can be calculated by the relation,

$$\tan\phi_1 = \frac{X_1}{R_1}, \tan\phi_2 = \frac{X_2}{R_2}, \ldots, \tan\phi_n = \frac{X_n}{R_n}$$

2) Voltage must be taken as reference phasor as it is common to all branches.
3) Represent all the currents on the phasor diagram and add them graphically or mathematically by expressing them in **rectangular form.** This is the resultant current drawn from the supply.
4) The phase angle of resultant current I is power factor angle. Cosine of this angle is the power factor of the circuit..

Concept of Admittance

Admittance is defined as the reciprocal of the impedance. It is denoted by Y and is measured in unit **siemens** or **mho.**

Now, current equation for the circuit shown in the Fig. 4.34 is,

$$\bar{I} = \bar{I_1} + \bar{I_2} + \bar{I_3}$$

$$\bar{I} = \bar{V} \times \left(\frac{1}{Z_1}\right) + \bar{V} \times \left(\frac{1}{Z_2}\right) + \bar{V} \times \left(\frac{1}{Z_3}\right)$$

$$\overline{VY} = \overline{VY_1} + \overline{VY_2} + \overline{VY_3}$$

$$\therefore \qquad \bar{Y} = \bar{Y_1} + \bar{Y_2} + \bar{Y_3}$$

where Y is the **admittance of the total circuit. The three impedances connected in parallel can be replaced by an equivalent circuit, where three admittances are connected in series,** as shown in the Fig. 4.34.

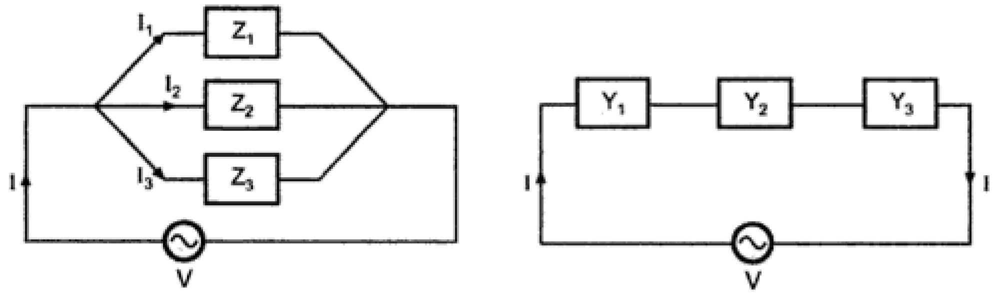


Fig. 4.34 Equivalent parallel circuit using admittances

Components of Admittance

Consider an impedance given as,

$$Z = R \pm j X$$

Positive sign for inductive and negative for capacitive circuit.

Admittance $\qquad Y = \frac{1}{Z} = \frac{1}{R \pm j X}$

Rationalising the above expression,

$$Y = \frac{R \mp jX}{(R \pm jX)(R \mp jX)} = \frac{R \mp jX}{R^2 + X^2}$$

$$= \left(\frac{R}{R^2 + X^2}\right) \mp j\left(\frac{X}{R^2 + X^2}\right) = \frac{R}{Z^2} \mp j\frac{X}{Z^2}$$

$\therefore \qquad \boxed{Y = G \mp jB}$

In the above expression, $\qquad G = \text{Conductance} = \frac{R}{Z^2}$

and $\qquad B = \text{Susceptance} = \frac{X}{Z^2}$

Conductance (G)

It is defined as the ratio of the resistance to the square of the impedance. It is measured in the unit **siemens.**

Susceptance (B)

It is defined as the ratio of the reactance to the square of the impedance. It is measured in the unit **siemens.**

Example 4.9 : *Two impedance $Z_1 = 5 - j13.1\ \Omega$ and $Z_1 = 8.57 + j6.42\ \Omega$ are connected in parallel across a voltage of (100 + j200) volts.*

Estimate :-

i) branch currents in complex form ii) total power consumed,

Draw a neat phasor diagram showing voltage, branch currents and all phase angles.

Solution : The circuit is shown in the Fig. 4.36.

$$V = 100 + j\,200 = 223.607 \angle 63.43° \text{ V}$$

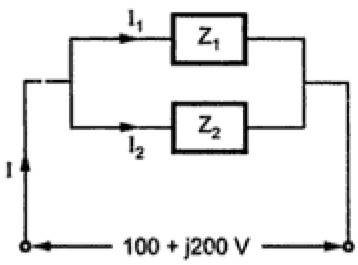


Fig. 4.36

$$Z_1 = 5 - j\,13.1 = 14.021 \angle -69.109°\ \Omega$$

$$Z_2 = 8.57 + j\,6.42 = 10.71 \angle +36.83°\ \Omega$$

i) $$I_1 = \frac{V}{Z_1} = \frac{223.607\angle 63.43°}{14.021\angle 69.109°}$$

$$= 15.948 \angle 132.539° \text{ A}$$

$$= \mathbf{-10.782 + j\,11.75\ A}$$

$$I_2 = \frac{V}{Z_2} = \frac{223.607\angle 63.43°}{10.71\angle +36.83°}$$

$$= 20.878 \angle 26.6° \text{ A} = \mathbf{18.668 + j\,9.3483\ A}$$

$$\therefore \quad I_T = \bar{I}_1 + \bar{I}_2 = -10.782 + j\,11.75 + 18.668 + j\,9.3483$$

$$= 7.886 + j\,21.0983 \text{ A} = \mathbf{22.5239 \angle 69.5° \ A}$$

$$\therefore \quad \phi_T = \text{Angle between V and } I_T$$

$$= 69.5 - 63.43 = \mathbf{6.075°} \text{ leading}$$

$$\therefore \quad P_T = V\,I_T \cos\phi_T = 223.607 \times 22.5239 \times \cos(6.075)$$

$$= \mathbf{5008.212\ W}$$

The phasor diagram is shown in the Fig. 4.37.

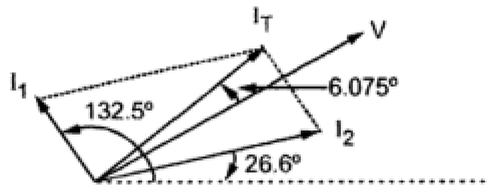


Fig. 4.37

Example 4.11 : *Find the current through 4 Ω resistor by using loop current method.*

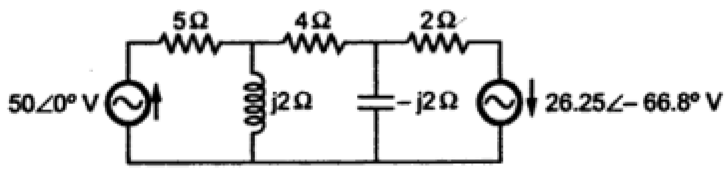


Fig. 4.39

Solution : The various loop currents are shown in the Fig. 4.39 (a),

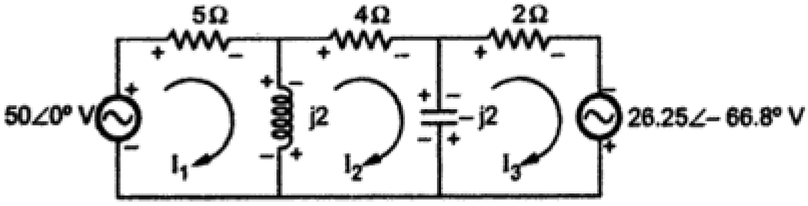


Fig. 4.39 (a)

Loop 1, $\quad -5I_1 - j2I_1 + j2I_2 + 50\angle 0^\circ = 0$

$\therefore \quad I_1(5+j2) - I_2\,j2 = 50\angle 0^\circ \qquad \ldots (1)$

Loop 2, $\quad -4I_2 - I_2(-j2) + I_3(-j2) - j2I_2 + j2I_1 = 0$

$\therefore \quad I_1(j2) + I_2(-4) - I_3(j2) = 0 \qquad \ldots (2)$

Loop 3, $\quad -2I_3 + 26.25\angle -66.8^\circ - I_3(-j2) + I_2(-j2) = 0$

$\therefore \quad I_2(j2) + I_3(2-j2) = 26.25\angle -66.8^\circ \qquad \ldots (3)$

$$\therefore \quad D = \begin{vmatrix} 5+j2 & -j2 & 0 \\ j2 & -4 & -j2 \\ 0 & j2 & 2-j2 \end{vmatrix}$$

$$= -4(2-j2)(5+j2) - 4(2-j2) - 4(5+j2)$$

$$= -84 + j24$$

and
$$D_2 = \begin{vmatrix} 5+j2 & 50\angle 0^\circ & 0 \\ j2 & 0 & -j2 \\ 0 & 26.25\angle -66.8^\circ & 2-j2 \end{vmatrix}$$

$$= -j2(2-j2)\,50\angle 0^\circ + j2(5+j2)(26.25\angle -66.8^\circ)$$

$$= [2\angle -90^\circ \times 2.828\angle -45^\circ \times 50\angle 0^\circ] + [2\angle 90^\circ \times 5.385\angle 21.8^\circ \times 26.25\angle -66.8^\circ]$$

$$= -199.969 - j199.969 + 199.9 + j199.9 = 0$$

$$\therefore \quad I_2 = \frac{D_2}{D} = \mathbf{0\ A} = \text{current through } 4\ \Omega$$

Example 4.12 : *Use the nodal analysis to find the value of V_x in the circuit shown in the Fig. 4.40 such that the current through (2 + j3) Ω impedance is zero.*

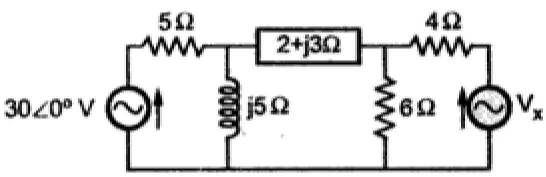


Fig. 4.40

Solution : The various node voltages and currents are shown in the Fig. 4.40 (a).

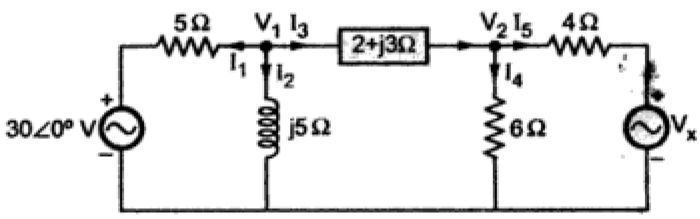


Fig. 4.40 (a)

At node 1, $$-I_1 - I_2 - I_3 = 0$$

$$\therefore \quad -\left[\frac{V_1 - 30\angle 0^\circ}{5}\right] - \left[\frac{V_1}{j5}\right] - \left[\frac{V_1 - V_2}{2+j3}\right] = 0$$

But current through (2 + j3) Ω i.e. $I_3 = 0$

$$\therefore \quad -\left[\frac{V_1 - 30\angle 0^\circ}{5}\right] - \frac{V_1}{j5} = 0$$

$$\therefore \quad \frac{-V_1}{j5} + \frac{30\angle 0^\circ}{5} - \frac{V_1}{5\angle 90^\circ} = 0$$

$$\therefore \quad V_1\,[0.2 + 0.2\angle -90^\circ] = 6\angle -0^\circ$$

$$\therefore \quad V_1 = \frac{6\angle 0^\circ}{0.2 - j0.2} = \frac{6\angle 0^\circ}{0.2828\angle -45^\circ}$$

$$= 21.216\angle +45^\circ \text{ V}$$

At node 2, $$I_3 - I_4 - I_5 = 0$$

i.e. $$I_4 + I_5 = 0 \qquad \text{... as } I_3 = 0$$

$$\therefore \quad \frac{V_2}{6} + \frac{V_2 - V_x}{4} = 0$$

$$\therefore \quad V_2\left[\frac{1}{6} + \frac{1}{4}\right] - \frac{V_x}{4} = 0$$

$$\therefore \quad 0.4166\,V_2 - 0.25\,V_x = 0$$

$$\therefore \quad V_x = 1.667\,V_2 \qquad \text{... (1)}$$

Example 4.13 : *Use the node voltage technique to obtain the current I in the network shown in the Fig. 4.41.*

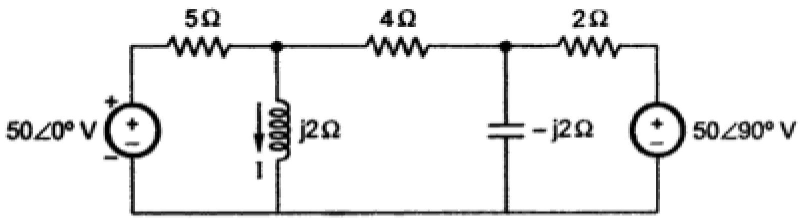


Fig. 4.41

As $I_3 = 0$, V_1 and V_2 must be equal.

$$\therefore \quad V_2 = V_1 = 21.2\angle 45^\circ \text{ V} \qquad \text{...(2)}$$

$$\therefore \quad V_x = 1.667 \times 21.2\angle +45^\circ = 35.33\angle 45^\circ \text{ V}$$

Solution : The various currents and node voltages are shown in the Fig. 4.41 (a).

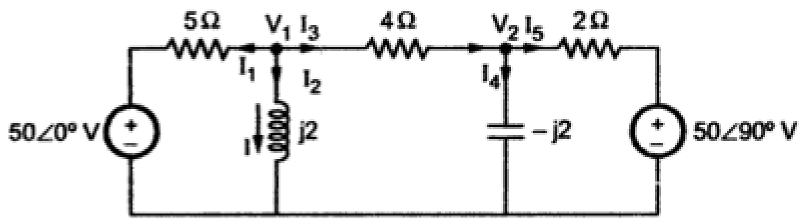


Fig. 4.41 (a)

At node 1, $$-I_1 - I_2 - I_3 = 0 \quad \text{i.e. } I_1 + I_2 + I_3 = 0$$

$$\therefore \quad \frac{V_1 - 50\angle 0^\circ}{5} + \frac{V_1}{j2} + \frac{V_1 - V_2}{4} = 0$$

$$\therefore \quad V_1\left[\frac{1}{5}-j\frac{1}{2}+\frac{1}{4}\right]-V_2\left[\frac{1}{4}\right] = 10 \qquad \ldots \frac{1}{j}=-j$$

$$\therefore \quad V_1\,[0.45-j0.5]-V_2\,(0.25) = 10 \qquad \ldots (1)$$

At node 2, $\quad I_3 - I_4 - I_5 = 0$

$$\therefore \quad \frac{V_1-V_2}{4}-\frac{V_2}{-j2}-\frac{V_2-50\angle 90^\circ}{2} = 0$$

$$\therefore \quad V_1\left[\frac{1}{4}\right]+V_2\left[-\frac{1}{4}-j\frac{1}{2}-\frac{1}{2}\right] = -25\angle 90^\circ \qquad \ldots \frac{1}{j}=-j$$

$$\therefore \quad 0.25\,V_1 + V_2\,[-0.75-j\,0.5] = -j25 \qquad \ldots (2)$$

For calculating $I = I_2$, only V_1 is required.

$$\therefore \quad D = \begin{vmatrix} 0.45-j0.5 & -0.25 \\ 0.25 & -0.75-j0.5 \end{vmatrix}$$

$$= -0.3375 + j\,0.375 - j\,0.225 - 0.25 + 0.0625$$

$$= -0.525 + j0.15 = 0.546\angle 164.05^\circ$$

$$D_1 = \begin{vmatrix} 10 & -0.25 \\ -j25 & -0.75-j0.5 \end{vmatrix} = -7.5 - j5 - j6.25$$

$$= -7.5 - j\,11.25 = 13.52\angle -123.69^\circ$$

$$\therefore \quad V_1 = \frac{D_1}{D} = \frac{13.52\angle -123.69^\circ}{0.546\angle 164.05^\circ} = \mathbf{24.762\angle 40.36^\circ\ V}$$

$$\therefore \quad I = \frac{V_1}{j2} = \frac{24.762\angle 40.36^\circ}{2\angle 90^\circ} = \mathbf{12.381\angle -49.64^\circ\ A}$$

Magnetically Coupled Circuits

When the two circuits are placed very close to each other such that a magnetic flux produced by one circuit links with both the circuits, then the two circuits are said to be **Magnetically Coupled Circuits.**

A wire of certain length, when twisted into coil becomes a basic inductor. If a current is made to pass through an inductor, an electromagnetic field is developed. A change in the magnitude of the current, changes the electromagnetic field and hence induces a voltage in coil according to Faraday's law of electromagnetic induction.

When two or more coils are placed very close to each other, then the current in one coil affects other coils by inducing voltage in them. Such coils are said to be **mutually coupled coils**. Such induced voltages in the coils are functions of the self inductances of the coils and mutual inductance between them. Let us study concept of self induced e.m.f. and mutually induced e.m.f.

Self inductance:

Consider a coil having N turns carrying current i as shown in the Fig. 2.1.

Due to the current flow, the flux ϕ is produced in the coil. The flux is measured in Wb (weber). The flux produced by the coil links with the coil itself. Thus the total flux linkage of the coil will be ($N\phi$) Wb-turns. If the current flowing through the coil changes, the flux produced in the coil also changes and hence flux linkage also changes.

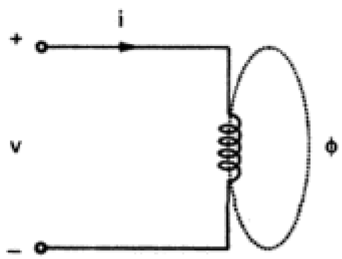


Fig. 2.1

According to Faraday's law, due to the rate of change of flux linkages, there will be induced e.m.f. in the coil. This phenomenon is called **self induction**. The e.m.f. or voltage induced in the coil due to the change of its own flux linked with it, is called **self induced e.m.f.**

According to Lenz's law the direction of this induced e.m.f. will be so as to oppose the cause producing it. The cause is the current I hence the self induced e.m.f. will try to set up a current which is in opposite direction to that of current I. When current is increased, self induced e.m.f. reduces the current tries to keep it to its original value. If current is decreased, self induced e.m.f. increases the current and tries to maintain it back to its original value. **So any change in current through coil is opposed by the coil.**

This property of the coil which opposes any change in the current passing through it is called **Self Inductance or Only Inductance.**

From the Faraday's law of electromagnetic induction, self induced e.m.f. can be expressed as

$$v = -N\frac{d\phi}{dt}$$

Negative sign indicates that direction of this e.m.f. is opposing change in current due to which it exists.

The flux can be expressed as,

$$\phi = (\text{Flux/ Ampere}) \times \text{Ampere} = \frac{\phi}{I} \times I$$

Now for a circuit, as long as permeability 'μ' is constant, ratio of flux to current (i.e. B/H) remains constant.

Rate of change of flux $= \frac{\phi}{I} \times$ Rate of change of current

$$\therefore \quad \frac{d\phi}{dt} = \frac{\phi}{I} \cdot \frac{dI}{dt}$$

$$v = -N \cdot \frac{\phi}{I} \cdot \frac{dI}{dt}$$

$$v = -\left(\frac{N\phi}{I}\right)\frac{dI}{dt}$$

The constant $\frac{N\phi}{I}$ in this expression is nothing but the quantitative measure of the property due to which coil opposes any change in current.

So this constant $\frac{N\phi}{I}$ is called **coefficient of self inductance and denoted by 'L'.**

$$\therefore \quad \boxed{L = \frac{N\phi}{I}}$$

It can be defined as flux linkages per ampere current in it. Its unit is **henry (H).**

A circuit possesses **a self inductance of 1 H** when a current of 1 A through it produces flux linkages of 1 Wb-turn in it.

$$\therefore \quad \boxed{v = -L\frac{dI}{dt} \quad \text{volts}}$$

Expressions for Coefficient of Self Inductance (L)

$$L = \frac{N\phi}{I}$$

But $\quad \phi = \frac{\text{M.M.F.}}{\text{Reluctance}} = \frac{NI}{S}$

$$\therefore \quad L = \frac{N \cdot NI}{I \cdot S}$$

$$\therefore \quad \boxed{L = \frac{N^2}{S} \quad \text{henries}}$$

Now $\quad S = \frac{l}{\mu a}$

$$L = \frac{N^2}{\left(\frac{l}{\mu a}\right)}$$

$$\therefore \quad \boxed{L = \frac{N^2 \mu a}{l} = \frac{N^2 \mu_0 \mu_r a}{l} \quad \text{henries}}$$

where $\quad l$ = Length of magnetic circuit

a = Area of cross-section of magnetic circuit through which flux is passing.

Example 1 : *If a coil has 500 turns is linked with a flux of 50 mWb, when carrying a current of 125 A. Calculate the inductance of the coil. If this current is reduced to zero uniformly in 0.1 sec, calculate the self induced e.m.f. in the coil.*

Solution : The inductance is given by,

$$L = \frac{N\phi}{I}$$

where $N = 500, \quad \phi = 50 \text{ mWb} = 50\times10^{-3} \text{ Wb}, \quad I = 25 \text{ A}$

$$\therefore \quad L = \frac{500\times50\times10^{-3}}{125} = \mathbf{0.2\ H}$$

$$v = -L\frac{dI}{dt}$$

$$= -L\left[\frac{\text{Final value of I} - \text{Initial value of I}}{\text{Time}}\right]$$

$$\therefore \quad v = -0.2\times\left(\frac{0-125}{0.1}\right) = \mathbf{250\ volts}$$

This is positive because current is decreased. So this 'v' will try to oppose this decrease, means will try to increase current and will help the growth of the current.

Mutually Induced E.M.F. and Mutual Inductance (M)

If the flux produced by one coil links with the other coil, placed sufficiently close to the first coil, then due to the change in the flux produced by first coil, there is induced e.m.f. in second coil. Such induced e.m.f. in the second coil is called **mutually induced e.m.f.**

Consider two coils which are placed very close to each other as shown in the Fig.

Let coil 1 has N_1 turns, while coil 2 has N_2 turns. The current flowing through coil 1 is i_1. Due to this current, the flux produced in coil 1 is ϕ_1. The part of this flux links with coil 2. This flux is called mutual flux.

It is denoted by ϕ_{21} as it is a part of flux ϕ_1 linking with coil 2. When current through coil 1 changes, the flux produced in coil 1 i.e. ϕ_1 changes. Thus flux associated with coil 2 i.e. ϕ_{21} changes. So according to the Faraday's law, there will be induced e.m.f. in coil 2.

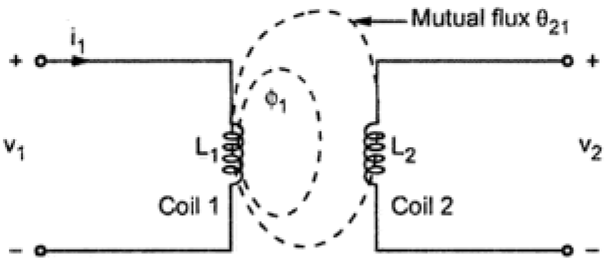


Fig.

Magnitude of Mutually Induced E.M.F.

Let N_1 = Number of turns of coil 1

N_2 = Number of turns of coil 2

I_1 = Current flowing through coil 1

ϕ_1 = Flux produced due to current I_1 in webers.

ϕ_2 = Flux linking with coil 2

According to Faraday's law, the induced e.m.f. in coil B is,

$$v_2 = -N_2\frac{d\phi_2}{dt}$$

Negative sign indicates that this e.m.f. will set up a current which will oppose the change of flux linking with it.

Now $$\phi_2 = \frac{\phi_2}{I_1} \times I_1$$

If permeability of the surroundings is assumed constant then $\phi_2 \propto I_1$ and hence ϕ_2 / I_1 is constant.

$$\therefore \quad \text{Rate of change of } \phi_2 = \frac{\phi_2}{I_1} \times \text{Rate of change of current } I_1$$

$$\therefore \quad \frac{d\phi_2}{dt} = \frac{\phi_2}{I_1} \cdot \frac{dI_1}{dt}$$

$$\therefore \quad v_2 = -N_2 \cdot \frac{\phi_2}{I_1} \cdot \frac{dI_1}{dt}$$

$$\therefore \quad v_2 = -\left(\frac{N_2 \phi_2}{I_1}\right) \frac{dI_1}{dt}$$

Here $\left(\frac{N_2 \phi_2}{I_1}\right)$ is called coefficient of mutual inductance denoted by M.

$$\therefore \quad \boxed{v_2 = -M\frac{dI_1}{dt} \quad \text{volts}}$$

Coefficient of mutual inductance is defined as the property by which e.m.f. gets induced in the second coil because of change in current through first coil.

Coefficient of mutual inductance is also called mutual inductance. It is measured in henries.

Coefficient of Coupling or Magnetic Coupling Coefficient (k)

Consider two coils having self inductances L_1 and L_2 placed very close to each other. Let the number of turns of the two coils be N_1 and N_2 respectively. Let coil 1 carries current i_1 and coil 2 carries current i_2.

Due to current i_1, the flux produced is ϕ_1 which links with both the coils. Then from the previous knowledge mutual inductance between two coils can be written as

$$M = \frac{N_1 \phi_{21}}{i_1} \quad \text{... (1)}$$

where ϕ_{21} is the part of the flux ϕ_1 linking with coil 2. Hence we can write, $\phi_{21} = k_1 \phi_1$.

$$\therefore \quad M = \frac{N_1 (k_1 \phi_1)}{i_1} \quad \text{... (2)}$$

Similarly due to current i_2, the flux produced is ϕ_2 which links with both the coils. Then the mutual inductance between two coils can be written as

$$M = \frac{N_2 \phi_{12}}{i_2} \quad \text{... (3)}$$

where ϕ_{12} is the part of the flux ϕ_2 linking with coil 1. Hence we can write $\phi_{12} = k_2 \phi_2$.

$$\therefore \quad M = \frac{N_2 (k_2 \phi_2)}{i_2} \quad \text{... (4)}$$

Multiplying equations (2) and (4),

$$M^2 = \frac{N_1 (k_1 \phi_1)}{i_1} \cdot \frac{N_2 (k_2 \phi_2)}{i_2}$$

$$\therefore \quad M^2 = k_1 k_2 \left[\frac{N_1 \phi_1}{i_1}\right]\left[\frac{N_2 \phi_2}{i_2}\right]$$

But $\quad \frac{N_1\phi_1}{i_1}$ = Self inductance of coil 1 = L_1

$\frac{N_2\phi_2}{i_2}$ = Self inductance of coil 2 = L_2

$$\therefore \quad M^2 = k_1 k_2 L_1 L_2$$

$$\therefore \quad M = \sqrt{k_1 k_2}\sqrt{L_1 L_2}$$

Let $\quad k = \sqrt{k_1 k_2}$

$$\therefore \quad \boxed{M = k\sqrt{L_1 L_2}} \qquad \text{... (5)}$$

where k is called **coefficient of coupling.**

$$\therefore \quad \boxed{k = \frac{M}{\sqrt{L_1 L_2}}} \qquad \text{... (6)}$$

Example 2 : *The number of turns in two coupled coils are 600 and 1200 respectively. When a current of 4 A flows in coil 1, the total flux in coil 1 is 0.5 mWb and the flux linking coil 2 is 0.4 mWb. Determine the self inductances of the coils and mutual inductance between them. Also calculate coefficient of coupling.*

Solution :

For coil 1, $\quad N_1 = 600$

$$i_1 = 4 \text{ A}$$

$$\phi_1 = 0.5 \text{ mWb}$$

$$\therefore \quad L = \frac{N_1\phi_1}{i_1} = \frac{(600)(0.5\times 10^{-3})}{4} = \mathbf{0.075\ H}$$

The self inductance of a coil is directly proportional to the square of the number of turns i.e. $L \propto N^2$.

$$\therefore \quad \frac{L_1}{L_2} = \frac{N_1^2}{N_2^2}$$

$$\therefore \quad L_2 = \left(\frac{N_2}{N_1}\right)^2 \cdot L_1 = \left(\frac{1200}{600}\right)^2 (0.075) = \mathbf{0.3\ H}$$

The flux linking with coil 2 is ϕ_{22} = 0.4 mWb

$$\therefore \quad M = \frac{N_2\phi_{21}}{i_1}$$

$$= \frac{(1200)\times(0.4\times 10^{-3})}{4}$$

$$= \mathbf{0.12\ H}$$

Hence the coefficient of coupling is given by,

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.12}{\sqrt{(0.075)(0.3)}} = \mathbf{0.8}$$

Dot Conventions

The sign of mutually induced voltage depends on direction of winding of the coils. But it is very inconvenient to supply the information about winding direction of the coils. Hence dot conventions are used for purpose of indicating direction of winding. The dot conventions are interpreted as below :

1. **If a current enters a dot in one coil, then mutually induced voltage in other coil is positive at the dotted end.**
2. **If a current leaves a dot in one coil, then mutually induced voltage in other coil is negative at the dotted end.**

Consider two magnetically coupled coils L_1 and L_2 wound on same core. Let current through coils L_1 and L_2 be i_1 and i_2 respectively. All the possible combinations of the dot convention between the magnetically coupled coils are as shown in the Fig. 2.4 (a), (c), (e) and (g). The equivalent circuits of all possible dot convention are as shown in the Fig. 2.4 (b), (d), (f) and (h) respectively.

Consider a magnetically coupled circuit with dots placed as shown in the Fig. 2.4 (a). Both the currents, i_1 and i_2 are entering the dotted terminals. Hence according to the dot convention, the mutually induced e.m.f. in both the coils has the polarity same as self

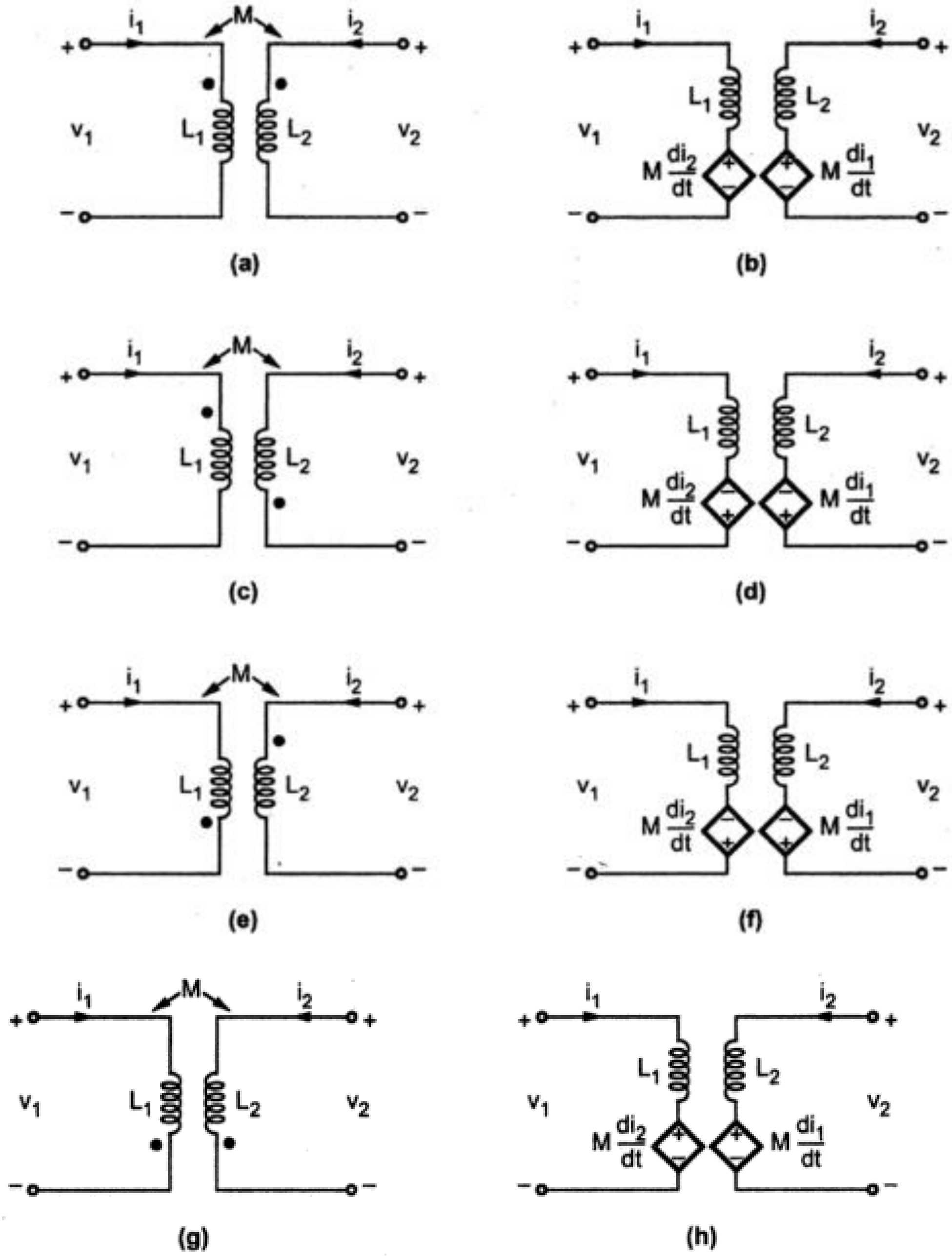


Fig. 2.4 Magnetically coupled circuits and equivalent circuits with different dot conventions

induced e.m.f. in respective coil. The equivalent circuit is as shown in the Fig. 2.4 (b). Applying KVL, the network equations of the equivalent circuit can be written as :

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \quad \text{... (1)}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \quad \text{... (2)}$$

Now consider magnetically coupled circuit as shown in the Fig. 2.4 (c) with dot placed at lower terminal of coil L_2. Hence current i_1 enters through dotted terminal of L_1 while current i_2 leaves through dotted terminal of L_2. So according to dot convention, the polarity of mutually induced e.m.f. in L_1 due to i_2 in L_2 will be opposite to that of self induced e.m.f. in coil L_1. Also the polarity of mutually induced e.m.f. in coil L_2 due to the current i_1 in coil L_1 will be opposite to that of self induced e.m.f. in coil L_2. The equivalent circuit is as shown in the Fig. 2.4 (d). By using KVL, the network equations can be written as,

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \quad \text{... (3)}$$

$$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} \quad \text{... (4)}$$

For the equivalent circuit shown in the Fig. 2.4 (f). Applying KVL, the network equations can be written as,

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \quad \text{... (5)}$$

$$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} \quad \text{... (6)}$$

For last possible combination, both the dots are placed at lower terminals of coils L_1 and L_2. Also both the currents leave dot as shown in the Fig. 2.4 (g). The equivalent circuit is as shown in the Fig. 2.4 (h). By applying KVL, the network equations can be written as,

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \quad \text{... (7)}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \quad \text{... (8)}$$

Uptill now we have discussed the coupled circuits in which two coils are magnetically coupled. But practically we may have to analyze a network with several windings.

The analysis of multiwinding inductor networks can be carried out for each pair of windings using same dot convention. In case of multiwinding inductor networks, the relationship between each pair of windings is represented by different forms of the dots such as ■ , ▲ , ● , ★ etc. The analysis of such multiwinding networks is illustrated in Example 2.4 and Example 2.5.

➠ **Example 3 :** *Calculate effective inductance of the circuit shown (Fig. 2.5) across terminals a and b.*

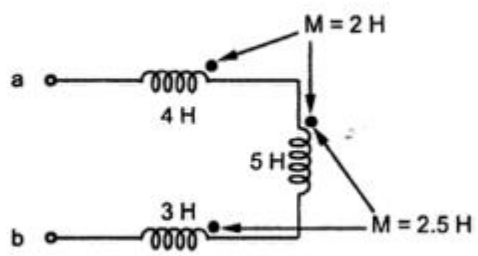


Fig. 2.5

Solution : Assume that current 'i' is flowing in series circuit and voltage developed across terminals a and b is shown in following Fig. 2.5 (a).

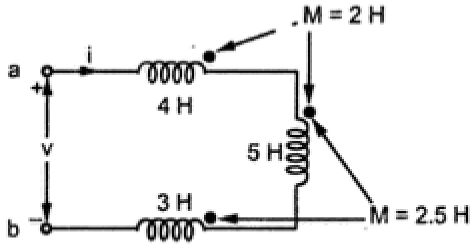


Fig. 2.5 (a)

Applying KVL for the above circuit. The current flowing through all the coils is same i.e. 'i'.

While writing the equations follow the convention that the current entering in the dot of one coil produces positive at the dotted end of the another coil while the current leaving from the dotted end of one coil produces the negative at the dotted end of the another coil.

$$-4\frac{di}{dt}+2\frac{di}{dt}-5\frac{di}{dt}+2\frac{di}{dt}-2.5\frac{di}{dt}-3\frac{di}{dt}-2.5\frac{di}{dt}+v=0$$

$$\therefore \quad v = 13\frac{di}{dt} = L_{eff}\frac{di}{dt}$$

$\therefore$ Effective inductance across terminals a and b is L_{eff}.

$$\therefore \quad L_{eff} = \mathbf{13\ H}$$

Inductive Coupling in Series

When two inductors having self inductances L_1 and L_2 are coupled in series, mutual inductance M exists between them. Two kinds of series connection are possible as follows.

Series Aiding

In this connection, two coils are connected in series such that their induced fluxes or voltages are additive in nature.

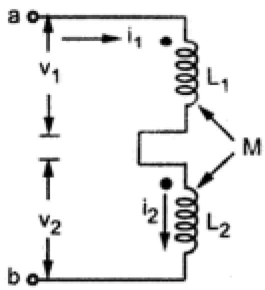


Fig. 2.8

Here currents i_1 and i_2 is nothing but current i which is entering dots for both the coils.

Self induced voltage in coil 1 = $v_1 = -L_1\frac{di}{dt}$

Self induced voltage in coil 2 = $v_2 = -L_2\frac{di}{dt}$

Mutually induced voltage in coil 1 due to change in current in coil 2 = $v_1' = -M\frac{di}{dt}$

Mutually induced voltage in coil 2 due to change in current in coil 1 = $v_2' = -M\frac{di}{dt}$

$\therefore$ Total induced voltage = $v_1 + v_2 + v_1' + v_2'$

$$= -\left(L_1\frac{di}{dt}+L_2\frac{di}{dt}+M\frac{di}{dt}+M\frac{di}{dt}\right)$$

$$= -(L_1 + L_2 + 2M)\frac{di}{dt}$$

If L is equivalent inductance across terminals a-b then total induced voltage in single inductance would be equal to $-L_{eff}\frac{di}{dt}$. Comparing two voltages,

$$\boxed{L_{eff} = L_1 + L_2 + 2M}$$

Series Opposing

In this connection, two coils are connected in such a way that, the induced fluxes or voltages are of opposite polarities.

Here i_1 and i_2 is same series current 'i' which is entering dot for coil L_1 and leaving dot for coil L_2.

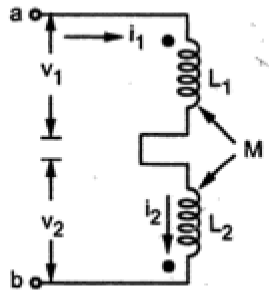


Fig. 2.9

Self induced voltage in coil 1 $= -L_1\dfrac{di}{dt}$

Self induced voltage in coil 2 $= -L_2\dfrac{di}{dt}$

Mutually induced voltage in coil 1 due to change in current in coil 2 $= v_1' = +M\dfrac{di}{dt}$

Also Mutually induced voltage in coil 2 due to change in current in coil 1 $= v_2' = +M\dfrac{di}{dt}$

Therefore total induced voltage $= v_1 + v_2 + v_1' + v_2'$

$$= -L_1\frac{di}{dt} - L_2\frac{di}{dt} + M\frac{di}{dt} + M\frac{di}{dt}$$

$$= -(L_1 + L_2 - 2M)\frac{di}{dt}$$

If L is equivalent inductance across terminals a and b then total induced voltage in single inductance would be equal to $-L_{eff}\dfrac{di}{dt}$. Comparing two voltages,

$$\boxed{L_{eff} = L_1 + L_2 - 2M}$$

Inductive Coupling in Parallel

When two inductors having self inductances L_1 and L_2 are coupled in parallel, we have two kinds of connections as follows.

Parallel Aiding

Consider parallel coupling of two inductors as shown in Fig. 2.10.

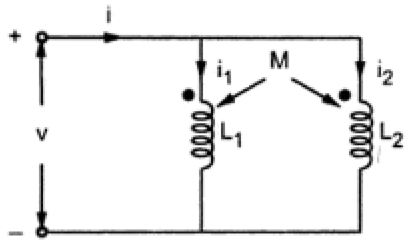


Fig. 2.10

Applying Kirchhoff's voltage law to both loops, we get,

$$-j\omega L_1 i_1 - j\omega M i_2 + v = 0$$

$$-j\omega L_2 i_2 - j\omega M i_1 + v = 0$$

i.e. $v = j\omega L_1 i_1 + j\omega M i_2$ ... (1)

$v = j\omega L_2 i_2 + j\omega M i_1$ ... (2)

We have, $j\omega L_1 \cdot i_1 + j\omega M \cdot i_2 = j\omega L_2 \cdot i_2 + j\omega M \cdot i_1$

But $i = i_1 + i_2$

i.e. $i_2 = i - i_1$

Putting value of i_2 in above equation, we get

$\therefore$ $j\omega L_1 i_1 + j\omega M(i - i_1) = j\omega L_2(i - i_1) + j\omega M i_1$

$\therefore$ $j\omega i_1(L_1 + L_2 - 2M) = j\omega i(L_2 - M)$

$$\therefore \qquad i_1 = \left[\frac{L_2 - M}{L_1 + L_2 - 2M}\right] i$$

Similarly,
$$i_2 = \left[\frac{L_1 - M}{L_1 + L_2 - 2M}\right] i$$

Putting values of i_1 and i_2 in equation (1), we get,

$$v = j\omega\left[\frac{L_1 (L_2 - M)}{L_1 + L_2 - 2M} + \frac{M (L_1 - M)}{L_1 + L_2 - 2M}\right] i$$

$$\therefore \qquad v = j\omega\left[\frac{L_1 L_2 - L_1 M + L_1 M - M^2}{L_1 + L_2 - 2M}\right] i$$

$$\therefore \qquad v = j\omega\left[\frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}\right] i \qquad \text{... (3)}$$

If L is effective inductance of parallel combination then,

$$v = j\omega L_{eff} \cdot i \qquad \text{... (4)}$$

Comparing equations (3) and (4) we have

$$\boxed{L_{eff} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}}$$

Parallel Opposing

Consider two inductors connected in parallel as shown in Fig. 2.11.

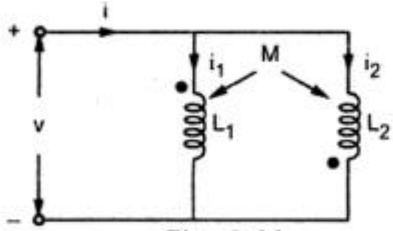


Fig. 2.11

Applying KVL to both loops, we get,

$$-j\omega L_1 i_1 + j\omega M i_2 + v = 0$$

$$-j\omega L_2 i_2 + j\omega M i_1 + v = 0$$

i.e. $j\omega L_1 i_1 - j\omega M i_2 = v$ ... (5)

$$j\omega L_2 i_2 - j\omega M i_1 = v \qquad \text{... (6)}$$

We have, $\quad j\omega L_1 i_1 - j\omega M i_2 = j\omega L_2 i_2 - j\omega M i_1$

But $\quad i = i_1 + i_2$

$\therefore \quad i_2 = i - i_1$

Substituting value of i_2 in above equation we have,

$$j\omega L_1 i_1 + j\omega M (i - i_1) = j\omega L_2 (i - i_1) - j\omega M i_1$$

$$\therefore \qquad j\omega i_1 (L_1 + L_2 + 2M) = j\omega i (L_2 + M)$$

$$\therefore \qquad i_1 = \left[\frac{L_2 + M}{L_1 + L_2 + 2M}\right] i$$

Similarly,
$$i_2 = \left[\frac{L_1 + M}{L_1 + L_2 + 2M}\right] i$$

Putting values of i_1 and i_2 in equation (5) we get,

$$v = j\omega\left[\frac{L_1 (L_2 + M)}{L_1 + L_2 + 2M} - \frac{M (L_1 + M)}{L_1 + L_2 + 2M}\right] i$$

$$= j\omega\left[\frac{L_1 L_2 + L_1 M - L_1 M - M^2}{L_1 + L_2 + 2M}\right] i$$

$$= j\omega\left[\frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}\right] i \qquad \text{... (7)}$$

If L is effective inductance of parallel combination,

$$v = j\omega L_{eff} \cdot i \qquad \dots (8)$$

Comparing equations (7) and (8) we have,

$$\boxed{L_{eff} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}}$$

Example : *If a coil of 800 μH is magnetically coupled to another coil of 200 μH. The coefficient of coupling between two coils is 0.05. Calculate inductance if two coils are connected in,*

(i) Series aiding (ii) Series opposing (iii) Parallel aiding (iv) Parallel opposing

Solution : The mutual inductance between two coils is given by

$$M = k \cdot \sqrt{L_1 L_2} = (0.05)\sqrt{(800\times10^{-6})(200\times10^{-6})}$$

$$= 20\ \mu H$$

Let the effective inductance for magnetically coupled coil be L.

(i) Series aiding :

$$L = L_1 + L_2 + 2M$$
$$= (800\times10^{-6})+(200\times10^{-6})+(2\times20\times10^{-6})$$
$$= \mathbf{1040\ \mu H}$$

(ii) Series opposing :

$$L = L_1 + L_2 - 2M$$
$$= (800\times10^{-6})+(200\times10^{-6})-(2\times20\times10^{-6})$$
$$= \mathbf{960\ \mu H}$$

(iii) Parallel aiding :

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$
$$= \frac{(800\times10^{-6}\times200\times10^{-6})-(20\times10^{-6})^2}{960\times10^{-6}}$$
$$= \frac{0.1596\times10^{-6}}{960\times10^{-6}} = \mathbf{166.25\ \mu H}$$

(iv) Parallel opposing :

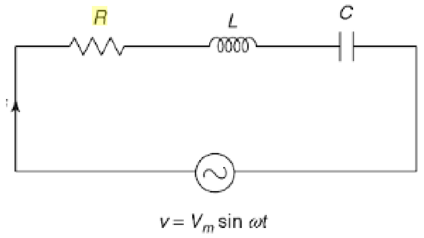
$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$
$$= \frac{(800\times10^{-6}\times200\times10^{-6})-(20\times10^{-6})^2}{1040\times10^{-6}}$$
$$= \frac{0.1596\times10^{-6}}{1040\times10^{-6}}$$
$$= \mathbf{153.46\ \mu H}$$

RESONANCE

Introduction:

Electrical resonance occurs in an electric circuit at a particular *resonant frequency* when the impedances or admittances of circuit elements cancel each other. In some circuits, this happens when the impedance between the input and output of the circuit is almost zero

SERIES RESONANCE



Series circuit

Resonance is a very important phenomenon in many electrical applications. The study of resonance is very useful in the telecommunication field. A circuit containing reactance is said to be in resonance if the voltage across the circuit is in phase with the current through it. At resonance, the circuit thus behaves as a pure resistor and the net reactance is zero.

As $X_L = 2\pi fL$. As frequency is changed from 0 to ∞, X_L increases linearily and graph of X_L against f is straight line passing through origin.

As $X_C = \frac{1}{2\pi fC}$, as frequency is changed from 0 to ∞, X_C reduces and the graph of X_C against f is rectangular hyperbola. Mathematically sign of X_C is opposite to X_L hence graph of X_L Vs f is shown in the first quadrant while X_C Vs f is shown in the third quadrant.

At $f = f_r$, the value of $X_L = X_C$ at this frequency.

As $X = X_L - X_C$, the graph of X against f is shown in the Fig. 4.1.

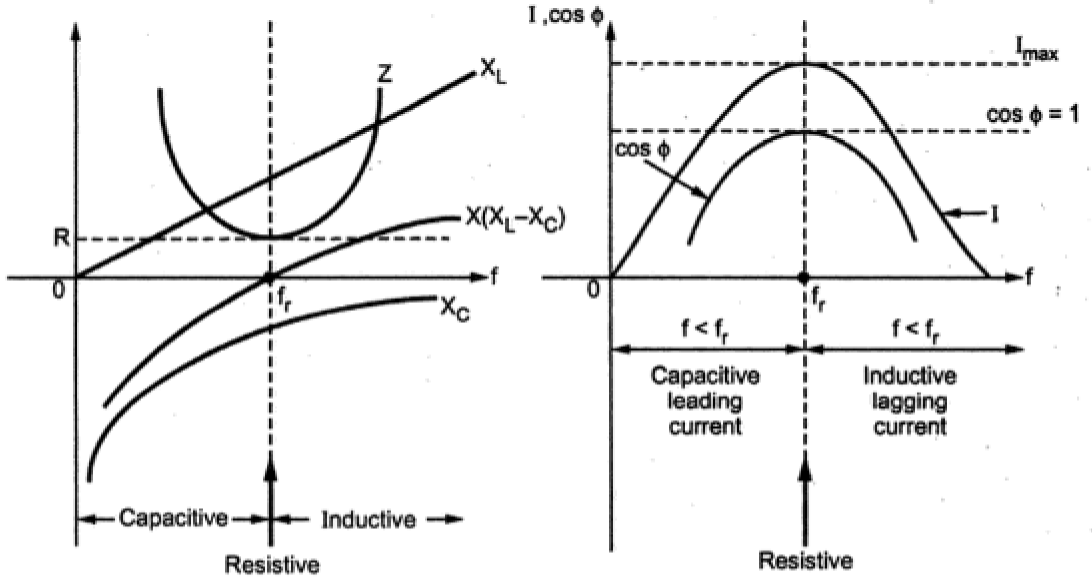


Fig. 4.1 Characteristics of series resonance

For $f < f_r$, the $X_C > X_L$ and net reactance X is **capacitive** while for $f > f_r$, the $X_L > X_C$ and net reactance X is **inductive**.

Now $Z = R + jX = R + j(X_L - X_C)$ but at $f = f_r$, $X_L = X_C$ and $X = 0$ hence the net impedance $Z = R$ which is purely resistive. So **impedance is minimum and purely resistive** at series resonance. The graph of Z against f is also shown in the Fig. 4.1.

Key Point : *As impedance is minimum, the current I = V/Z is maximum at series resonance.*

Now power factor $\cos\phi = R/Z$ and at $f = f_r$ as $Z = R$, the **power factor is unity** and at its **maximum** at series resonance. For **$f < f_r$** it is **leading** in nature while for **$f > f_r$** it is **lagging** in nature.

Resonant Frequency

Let f_r be the resonant frequency in Hz at which,

$$X_L = X_C$$

$$\therefore \quad 2\pi f_r L = \frac{1}{2\pi f_r C}$$

$$\therefore \quad (f_r)^2 = \frac{1}{4\pi^2 LC}$$

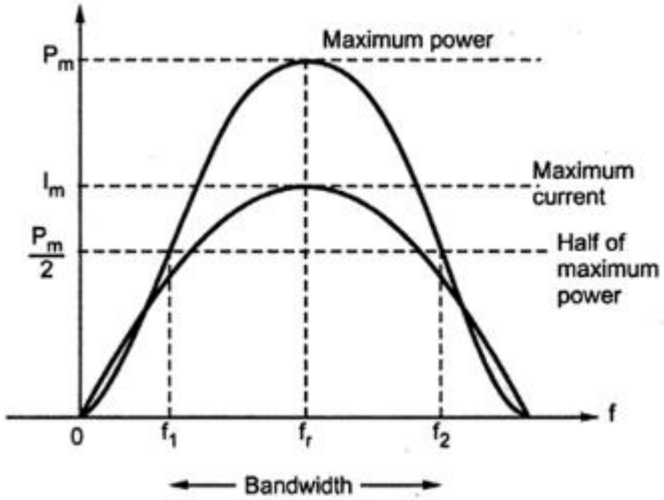
$$\therefore \quad f_r = \frac{1}{2\pi\sqrt{LC}} \text{ Hz}$$

i.e.
$$\boxed{\omega_r = \frac{1}{\sqrt{LC}} \text{ rad/sec}}$$

Bandwidth of Series R-L-C Circuit

At series resonance, current is maximum and impedance Z is minimum. Now power consumed in a circuit is proportional to square of the current as $P = I^2R$. So at series resonance as current is maximum, power is also at its maximum i.e. P_m. The Fig. 4.2 shows the graph of current and power against frequency.

It can be observed that at two frequencies f_1 and f_2 the power is half of its maximum value. These frequencies are called **half power frequencies**.



The difference between the half power frequencies f_1 and f_2 at which power is half of its maximum is called bandwidth of the series R-L-C circuit.

$\therefore \quad$ B.W. $= f_2 - f_1$

Expressions for Lower and Upper Cut-off Frequencies

The current in a series RLC circuit is given by the equation,

$$I = \frac{V}{Z} \qquad \text{but } Z = R + j\,(X_L - X_C)$$

$$\therefore \qquad I = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad \text{... (1)}$$

At resonance, $I_m = \frac{V}{R}$ (maximum value) ... (2)

and $P_m = I_m^2 R$

At half power point, $P = \frac{P_m}{2} = \frac{I_m^2}{2} R = \left(\frac{I_m}{\sqrt{2}}\right)^2 R$

$$\therefore \qquad I = \frac{I_m}{\sqrt{2}} \quad \text{at half power frequency}$$

Equating equations (1) and (2),

$$\frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V}{\sqrt{2}\cdot R}$$

$$\therefore \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}\, R$$

$$\therefore \qquad R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2\, R^2$$

$$\therefore \qquad \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$\therefore \qquad \omega L - \frac{1}{\omega C} = \pm R \qquad \text{...(3)}$$

From the equation (3) we can find two values of half power frequencies which are ω_1 and ω_2 corresponding to f_1 and f_2.

$$\therefore \qquad \omega_2 L - \frac{1}{\omega_2 C} = +R \qquad \text{...(4)}$$

$$\text{and} \qquad \omega_1 L - \frac{1}{\omega_1 C} = -R \qquad \text{... (5)}$$

$$\therefore \quad (\omega_1 + \omega_2)\, L - \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\frac{1}{C} = 0 \qquad \text{... Adding equations (4) and (5)}$$

$$\therefore \qquad (\omega_1 + \omega_2)\, L = \frac{(\omega_1 + \omega_2)}{\omega_1\, \omega_2}\cdot\frac{1}{C}$$

$$\therefore \qquad \omega_1\, \omega_2 = \frac{1}{LC} \qquad \text{... (6)}$$

$$\text{but} \qquad \omega_r = \frac{1}{\sqrt{LC}}$$

$$\therefore \qquad \boxed{\omega_1\, \omega_2 = (\omega_r)^2}$$

$$\therefore \qquad \boxed{f_1 f_2 = (f_r)^2} \qquad \text{... (7)}$$

The equation (7) shows that the **resonant frequency is the geometric mean of the two half power frequencies.**

$$\therefore \qquad \boxed{f_r = \sqrt{f_1 f_2}} \qquad \dots (8)$$

Subtracting equation (5) from equation (4) we get,

$$(\omega_2 - \omega_1)\, L - \left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right)\frac{1}{C} = 2R$$

$$\therefore \qquad (\omega_2 - \omega_1) + \frac{(\omega_2 - \omega_1)}{\omega_1\, \omega_2} \cdot \frac{1}{LC} = \frac{2R}{L} \qquad \dots \text{Dividing both sides by L}$$

$$\therefore \qquad (\omega_2 - \omega_1) + (\omega_2 - \omega_1) = \frac{2R}{L} \qquad \dots \text{As } \frac{1}{\omega_1 \omega_2} = LC$$

$$\therefore \qquad (\omega_2 - \omega_1) = \frac{R}{L}$$

$$\text{i.e.} \qquad f_2 - f_1 = \frac{R}{2\pi L} \qquad \dots (9)$$

$$\text{Thus} \qquad \boxed{B.W. = \frac{R}{2\pi L}}$$

The bandwidth is also denoted as,

B.W. = $2\Delta f$ where

$$\boxed{\Delta f = \frac{R}{4\pi L}} \qquad \text{as shown in the Fig. 4.3}$$

From Fig. 4.3 we can write,

$$\boxed{\begin{aligned} f_1 &= f_r - \Delta f \\ \text{and} \quad f_2 &= f_r + \Delta f \end{aligned}}$$

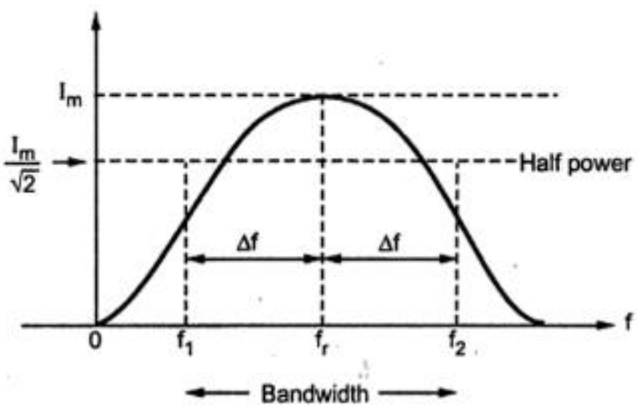


Fig. 4.3

Quality Factor

The quality factor of R-L-C series circuit is the voltage magnification in the circuit at resonance.

$$\text{Voltage magnification} = \frac{\text{Voltage across L or C}}{\text{Supply voltage}}$$

Now $\quad V_L = \text{Voltage across } L = I_m X_L = I_m \omega_r L \quad \text{at resonance}$

and $\quad I_m = \frac{V}{R} \quad \text{at resonance}$

$\therefore \quad V_L = \frac{V\omega_r L}{R} \quad \text{at resonance}$

$$\therefore \quad \text{Voltage magnification} = \frac{\frac{V\omega_r L}{R}}{V} = \frac{\omega_r L}{R}$$

This is nothing but quality factor Q.

$\therefore \quad \boxed{Q = \frac{\omega_r L}{R}} \qquad \text{but } \omega_r = \frac{1}{\sqrt{LC}}$

$\therefore \quad \boxed{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$

and $\quad \boxed{Q = \frac{\omega_r}{B.W.}} \qquad \text{as B.W.} = (\omega_2 - \omega_1) = \frac{R}{L}$

Example 1 : *A RLC series circuit with a resistance of 10 Ω, impedance of 0.2 H and a capacitance of 40μF is supplied with a 100 V supply at variable frequency. Find the following w.r.t the series resonant circuit :-*

i) the frequency at resonance ii) the current iii) power iv) power factor v) voltage across R, L, C at that time vi) quality factor of the circuit vii) half power points viii) phasor diagram.

Solution : The given values are, $R = 10\ \Omega$, $L = 0.2$ H, $C = 40\ \mu F$ and $V = 100$ V

i) $\quad f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 40 \times 10^{-6}}}$

$\quad = \mathbf{56.2697\ Hz}$

ii) $\quad I_m = \frac{V}{R} = \frac{100}{10} = \mathbf{10\ A} \qquad$ ... Current is maximum at resonance

iii) $\quad P_m = I_m^2 R = (10)^2 \times 10 = \mathbf{1000\ W}$

iv) Power factor is **unity**, as impedance is **purely resistive** at resonance

v) $\quad V_R = I_m R = 10 \times 10 = \mathbf{100\ V}$

$\quad X_L = 2\pi f_r L = 2\pi \times 56.2697 \times 0.2 = 70.7105\ \Omega$

$\therefore \quad V_L = I_m X_L = 10 \times 70.7105 = \mathbf{707.105\ V}$

and $\quad X_C = \frac{1}{2\pi f_r C}$

$\quad = \frac{1}{2\pi \times 56.2697 \times 40 \times 10^{-6}}$

$\quad = 70.7105\ \Omega$

$$\therefore \qquad \boxed{V_C = I_m X_C = 707.105 \text{ V}}$$

Thus $V_L = V_C$ at resonance

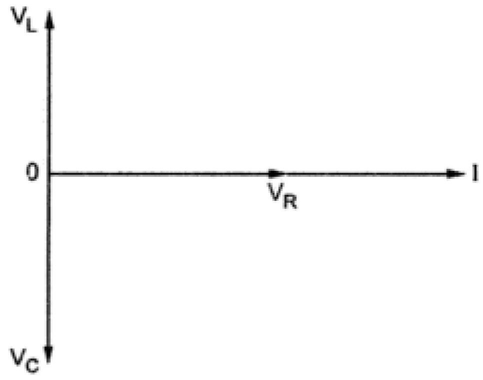
vi) $$Q = \frac{\omega_r L}{R} = \frac{2\pi f_r L}{R} = 7.071$$

vii) $$\Delta f = \frac{R}{4\pi L} = \frac{10}{4\pi \times 0.2} = 3.9788$$

$$\therefore \quad f_1 = f_r - \Delta f = 56.2697 - 3.9788 = 52.2909 \text{ Hz}$$

and $$f_2 = f_r + \Delta f = 56.2697 + 3.9788 = 60.2485 \text{ Hz}$$

viii) $$\text{B.W.} = f_2 - f_1 = 60.2485 - 52.2909 = 7.9576 \text{ Hz}$$



Resonance in Parallel Circuit

Similar to a series a.c. circuit, there can be a resonance in parallel a.c. circuit. When the power factor of a parallel a.c. circuit is unity i.e. the voltage and total current are in phase at a particular frequency then the parallel circuit is said to be at **resonance**. The frequency at which the parallel resonance occurs is called **resonant frequency** denoted as f_r Hz.

4.3.1 Characteristics of Parallel Resonance

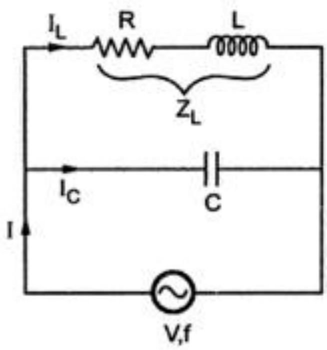


Fig. 4.5 Practical parallel circuit

Consider a practical parallel circuit used for the parallel resonance as shown in the Fig. 4.5.

The one branch consists of resistance R in series with inductor L. So it is series R-L circuit with impedance Z_L. The other branch is pure capacitive with a capacitor C. Both the branches are connected in parallel across a variable frequency constant voltage source.

The current drawn by inductive branch is I_L while drawn by capacitive branch is I_C.

$$I_L = \frac{V}{Z_L} \quad \text{where } Z_L = R + j X_L$$

and $$I_C = \frac{V}{X_C} \quad \text{where } X_C = \frac{1}{2\pi f C}$$

The current I_L lags voltage V by angle ϕ_L which is decided by R and X_L while the current I_C leads voltage V by 90°. The total current I is phasor addition of I_L and I_C. The phasor diagram is shown in the Fig. 4.5 (a).

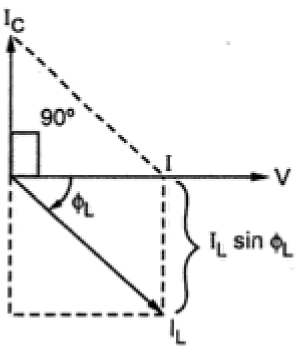


Fig. 4.5(a)

For the parallel resonance V and I must be in phase. To achieve this unity p.f. condition,

$$\boxed{\begin{aligned} I &= I_L \cos\phi_L \\ \text{and} \quad I_C &= I_L \sin\phi_L \end{aligned}}$$

From the impedance triangle of R-L series circuit we can write,

$$\tan\phi_L = \frac{X_L}{R}, \ \cos\phi_L = \frac{R}{Z_L}, \ \sin\phi_L = \frac{X_L}{Z_L}$$

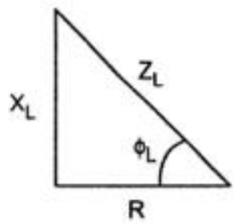
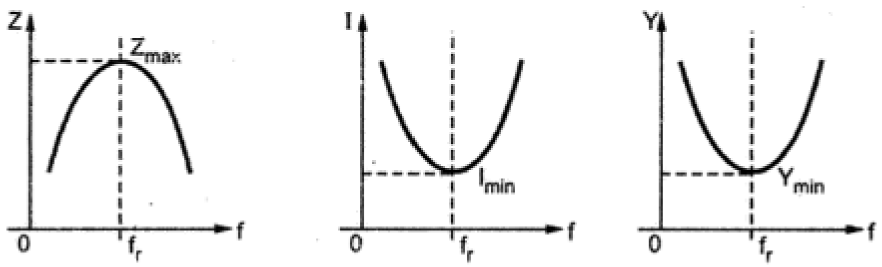


Fig. 4.5(b) Impedance triangle

As frequency is increased, $X_L = 2\pi f L$ increases due to which $Z_L = \sqrt{R^2 + X_L^2}$ also increases. Hence $\cos\phi_L$ decreases and $\sin\phi_L$ increases. As Z_L increases, the current I_L also decreases.

At resonance $f = f_r$ and $I_L \cos\phi_L$ is at its minimum. Thus at resonance **current is minimum** while the total **impedance** of the circuit is **maximum**. As admittance is reciprocal of impedance, as frequency is changed, **admittance** decreases and is **minimum** at resonance. The three curves are shown in the Fig. 4.6 (a), (b) and (c).



(a) Impedance (b) Current (c) Admittance

Fig. 4.6 Characteristics of parallel resonance

Expression for Resonant Frequency

At resonance $I_C = I_L \sin\phi_L$

$$\therefore \quad \frac{V}{X_C} = \frac{V}{Z_L}\cdot\frac{X_L}{Z_L} = \frac{VX_L}{Z_L^2}$$

$$\therefore \quad Z_L^2 = X_L X_C$$

$$\therefore \quad R^2 + (2\pi f_r L)^2 = (2\pi f_r L)\times\frac{1}{2\pi f_r C} \quad \text{as } f = f_r$$

$$\therefore \quad R^2 + (2\pi f_r L)^2 = \frac{L}{C}$$

$$\therefore \quad (2\pi f_r L)^2 = \frac{L}{C} - R^2$$

$$\therefore \quad (2\pi f_r)^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$\therefore \quad \boxed{f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}}$$

Thus if R is very small compared to L and C, $\frac{R^2}{L^2} << \frac{1}{LC}$

$$\therefore \quad \boxed{f_r = \frac{1}{2\pi\sqrt{LC}}}$$

... Neglecting effect of R

Dynamic Impedance at Resonance

The impedance offered by the parallel circuit at resonance is called **dynamic impedance** denoted as Z_D. This is maximum at resonance. As current drawn at resonance is minimum, the parallel circuit at resonance is called **rejector circuit**. This indicates that it rejects the unwanted frequencies and hence it is used as filter in radio receiver.

From $I_C = I_L \sin \phi_L$ we have seen that,

$$Z_L^2 = \frac{L}{C}$$

While
$$I = I_L \cos \phi_L = \frac{V}{Z_L} \cdot \frac{R}{Z_L}$$

$$= \frac{VR}{Z_L^2}$$

$$\therefore \quad I = \frac{VR}{\frac{L}{C}}$$

$$= \frac{V}{(L/RC)}$$

$$\therefore \quad I = \frac{V}{Z_D}$$

where
$$\boxed{Z_D = \frac{L}{RC} = \text{Dynamic impedance}}$$

Quality Factor of Parallel Circuit

The parallel circuit is used to magnify the current and hence known as current resonance circuit.

The quality factor of the parallel circuit is defined as the current magnification in the circuit at resonance.

The current magnification is defined as,

$$\text{Current magnification} = \frac{\text{Current in the inductive branch}}{\text{Current in supply at resonance}} = \frac{I_L}{I}$$

$$= \frac{\frac{V}{Z_L}}{\frac{V}{Z_D}} = \frac{Z_D}{Z_L}$$

$$= \frac{\frac{L}{RC}}{\sqrt{\frac{L}{C}}} = \frac{1}{R}\sqrt{\frac{L}{C}} \quad \text{as } Z_L$$

$$= \sqrt{X_L X_C} = \sqrt{\frac{L}{C}}$$

This is nothing but the quality factor at resonance.

$$\therefore \quad \boxed{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

Example 2 : *An inductive coil of resistance 10 Ω and inductance 0.1 Henrys is connected in parallel with a 150 μF capacitor to a variable frequency, 200 V supply. Find the resonant frequency at which the total current taken from the supply is in phase with the supply voltage. Also find the value of this current. Draw the phasor diagram.*

Solution : The circuit is shown in the Fig. 4.7.

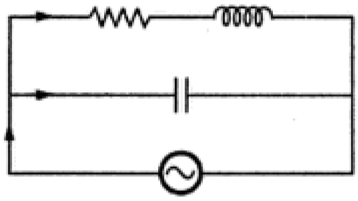


Fig. 4.7

The resonant frequency is,

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1\times 150\times 10^{-6}} - \frac{(10)^2}{(0.1)^2}}$$

$$= \mathbf{37.8865 \quad Hz}$$

Now $\quad Z_L = R + j X_L = 10 + j\ (2\pi f_r L)$

$$= 10 + j\,23.805 = 25.82 \angle 67.21° \ \Omega$$

$\therefore \quad I_L = \frac{V}{Z_L} = \frac{200\angle 0°}{25.82\angle 67.21°} = \mathbf{7.7459 \angle -67.21° \ A}$

and $\quad I_C = \frac{V}{X_C} = \frac{200\angle 0°}{\frac{1}{2\pi f_r C}\angle -90°} = \frac{200\angle 0°}{28\angle -90°} = \mathbf{7.143 \angle +90° \ A}$

where $\quad Z_C = 0 - j X_C = 0 - j\,28 = 28 \angle -90° \ \Omega$

$\therefore \quad Z_T = \frac{Z_C Z_L}{Z_C + Z_L} = \frac{28\angle -90° \times 25.82\angle 67.21°}{0 - j\,28 + 10 + j\,23.805}$

$$= \frac{722.96\angle -22.79°}{10 - j\,4.195}$$

$$= \frac{722.96\angle -22.79}{10.844\angle -22.79}$$

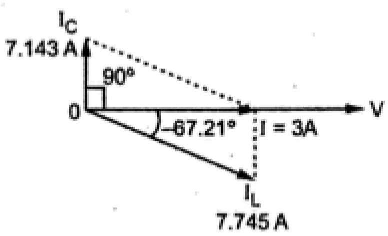
$$= \mathbf{66.67 \ \Omega \ pure \ resistive}$$

$\therefore \quad Z_T = Z_D = \frac{L}{CR}$

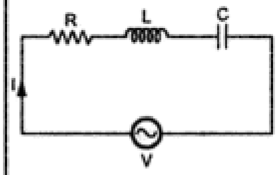
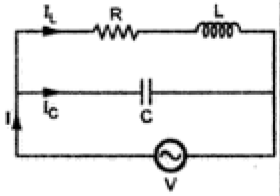
$$= \frac{0.1}{150\times 10^{-6}\times 10}$$

$$= 66.67 \ \Omega$$

$\therefore \quad I = \frac{V}{Z_D} = \frac{200}{66.67} = \mathbf{3 \ A}$



Comparison of Resonant Circuits

Sr. No.	Parameter	Series Resonant	Parallel Resonant
1.	Circuit		
2.	Type of circuit	Purely resistive	Purely resistive
3.	Power factor	Unity	Unity
4.	Impedance	Minimum Z = R	Dynamic but maximum $Z_D = \frac{L}{RC}$
5.	Frequency	$f_r = \frac{1}{2\pi\sqrt{LC}}$	$f_r = \frac{1}{2\pi\sqrt{LC}}$
6.	Current	Maximum $I = \frac{V}{R}$	Minimum $I = \frac{V}{Z_D}$
7.	Magnification	Voltage magnification	Current magnification
8.	Quality factor	$Q = \frac{\omega_r L}{R} = \frac{\omega_r}{B.W.}$	$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$
9.	Nature	Acceptor	Rejector
10.	Practical used	Radio circuits sharpness of tunning circuit	Impedance for matching, tuning, as a filter

UNIT – IV

NETWORK THEOREMS

SUPERPOSITION THEOREM

It states that '*in a linear network containing more than one independent source and dependent source, the resultant current in any element is the algebraic sum of the currents that would be produced by each independent source acting alone, all the other independent sources being represented meanwhile by their respective internal resistances.*'

The independent voltage sources are represented by their internal resistances if given or simply with zero resistances, i.e., short circuits if internal resistances are not mentioned. The independent current sources are represented by infinite resistances, i.e., open circuits.

The dependent sources are not sources but dissipative components—hence they are active at all times. A dependent source has zero value only when its control voltage or current is zero.

A linear network is one whose parameters are constant, i.e., they do not change with voltage and current.

Explanation Consider the network shown in Fig. 3.1. Suppose we have to find current I_4 through resistor R_4.

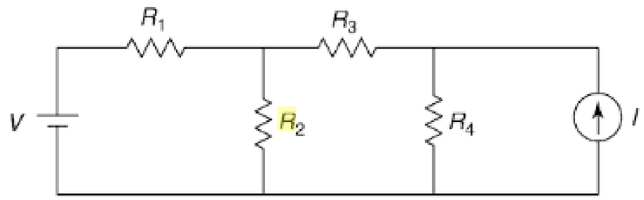


Fig. 3.1 *Network to illustrate superposition theorem*

The current flowing through resistor R_4 due to constant voltage source V is found to be say I_4' (with proper direction), representing constant current source with infinite resistance, i.e., open circuit.

The current flowing through resistor R_4 due to constant current source I is found to be say I_4'' (with proper direction), representing the constant voltage source with zero resistance or short circuit.

The resultant current I_4 through resistor R_4 is found by superposition theorem.

$$I_4 = I_4' + I_4''$$

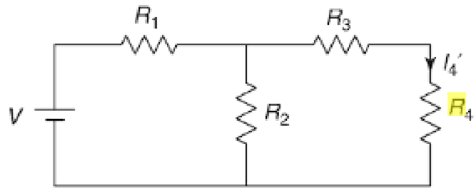


Fig. 3.2 *When voltage source V is acting alone*

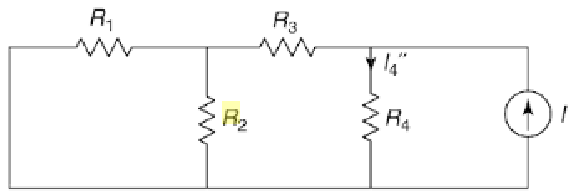


Fig. 3.3 *When current source I is acting alone*

Steps to be followed in Superposition Theorem

1. Find the current through the resistance when only one independent source is acting, replacing all other independent sources by respective internal resistances.
2. Find the current through the resistance for each of the independent sources.
3. Find the resultant current through the resistance by the superposition theorem considering magnitude and direction of each current.

Example 3.1 *Find the current through the 2 Ω resistor in Fig. 3.4.*

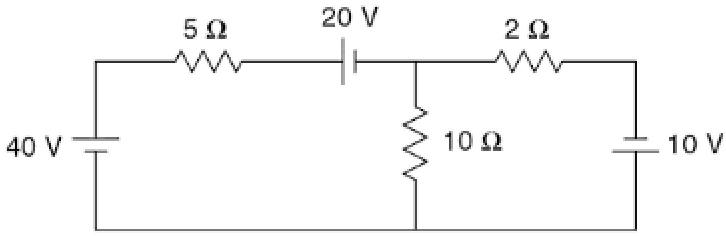


Fig. 3.4

Solution

Step I When the 40 V source is acting alone (Fig. 3.5)

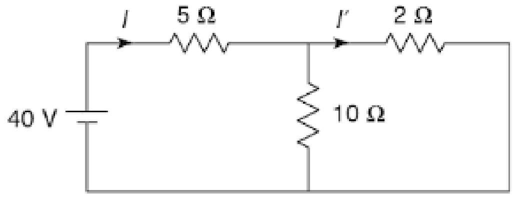


Fig. 3.5

By series parallel reduction technique (Fig. 3.6),

$$I = \frac{40}{5+1.67} = 6 \text{ A}$$

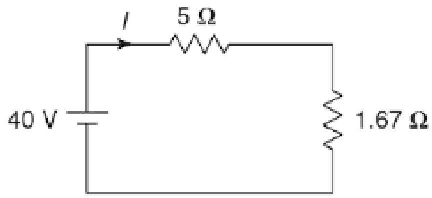


Fig. 3.6

From Fig. 3.5, by current-division rule,

$$I' = 6 \times \frac{10}{10+2} = 5 \text{ A } (\rightarrow)$$

Step II When the 20 V source is acting alone (Fig. 3.7)

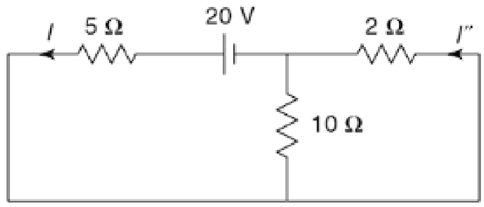


Fig. 3.7

By series–parallel reduction technique (Fig. 3.8),

$$I = \frac{20}{5+1.67} = 3 \text{ A}$$

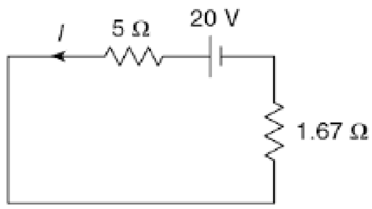


Fig. 3.8

From Fig. 3.7, by current-division rule,

$$I'' = 3 \times \frac{10}{10+2} = 2.5 \text{ A} (\leftarrow) = -2.5 \text{ A} \quad (\rightarrow)$$

Step III When the 10 V source is acting alone (Fig. 3.9)

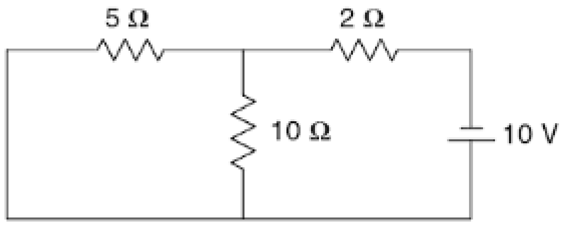


Fig. 3.9

By series–parallel reduction technique (Fig. 3.10),

$$I''' = \frac{10}{3.33+2} = 1.88\text{ A}(\rightarrow)$$

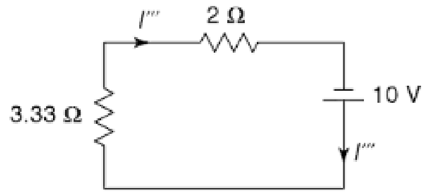


Fig. 3.10

Step IV By superposition theorem,

$$I = I' + I'' + I''' = 5 - 2.5 + 1.88 = 4.38\text{ A}\quad(\rightarrow)$$

THEVENIN'S THEOREM

It states that '*any two terminals of a network can be replaced by an equivalent voltage source and an equivalent series resistance. The voltage source is the voltage across the two terminals with load, if any, removed. The series resistance is the resistance of the network measured between two terminals with load removed and constant voltage source being replaced by its internal resistance (or if it is not given with zero resistance, i.e., short circuit) and constant current source replaced by infinite resistance, i.e., open circuit.*'

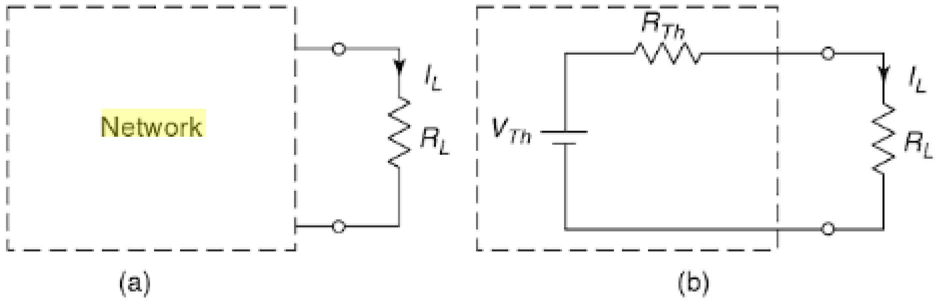


Fig. 3.109 *Network illustrating Thevenin's theorem*

Explanation Consider a simple network as shown in Fig. 3.110.

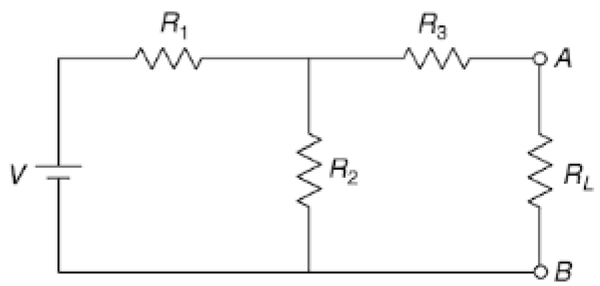


Fig. 3.110 *Network*

For finding load current through R_L, first remove the load resistor R_L from the network and calculate open circuit voltage V_{Th} across points A and B as shown in Fig. 3.111.

$$V_{Th} = \frac{R_2}{R_1 + R_2}V$$

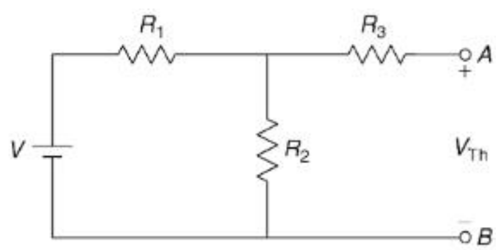


Fig. 3.111 *Calculation of V_{Th}*

For finding series resistance R_{Th}, replace the voltage source by a short circuit and calculate resistance between points A and B as shown in Fig. 3.112.

$$R_{Th} = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

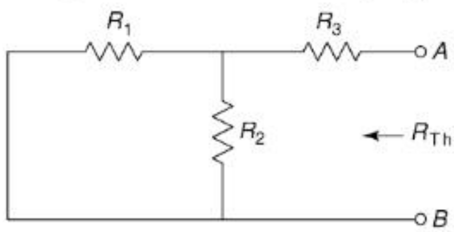


Fig. 3.112 *Calculation of R_{Th}*

Thevenin's equivalent network is shown in Fig. 3.113.

$$I_L = \frac{V_{Th}}{R_{Th} + R_L}$$

If the network contains both independent and dependent sources, Thevenin's resistance R_{Th} is calculated as,

$$R_{Th} = \frac{V_{Th}}{I_N}$$

where I_N is the short-circuit current which would flow in a short circuit placed across the terminals A and B. Dependent sources are active at all times. They have zero values only when the control voltage or current is zero. R_{Th} may be negative in some cases which indicates negative resistance region of the device, i.e., as voltage increases, current decreases in the region and vice-versa.

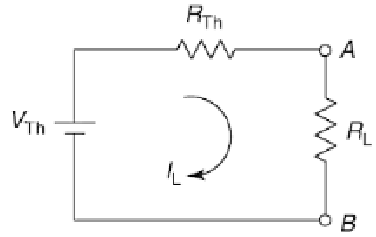


Fig 3.113 *Thevenin's equivalent network*

If the network contains only dependent sources then

$$V_{Th} = 0$$

$$I_N = 0$$

For finding R_{Th} in such a network, a known voltage V is applied across the terminals A and B and current is calculated through the path AB.

$$R_{Th} = \frac{V}{I}$$

or a known current source I is connected across the terminals A and B and voltage is calculated across the terminals A and B.

$$R_{Th} = \frac{V}{I}$$

Thevenin's equivalent network for such a network is shown in Fig. 3.114.

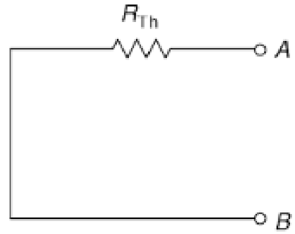


Fig. 3.114 *Thevenin's equivalent network*

Steps to be Followed in Thevenin's Theorem

1. Remove the load resistance R_L.
2. Find the open circuit voltage V_{Th} across points A and B.
3. Find the resistance R_{Th} as seen from points A and B.
4. Replace the network by a voltage source V_{Th} in series with resistance R_{Th}.
5. Find the current through R_L using Ohm's law.

$$I_L = \frac{V_{Th}}{R_{Th} + R_L}$$

Example 3.27 *Find the current through the 2 Ω resistor in Fig. 3.115.*

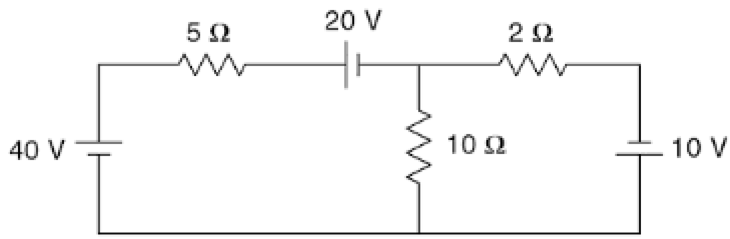


Fig. 3.115

Solution

Step I Calculation of V_{Th} (Fig. 3.116)
Applying KVL to the mesh,

$$40 - 5I - 20 - 10I = 0$$

$$15I = 20$$

$$I = 1.33 \text{ A}$$

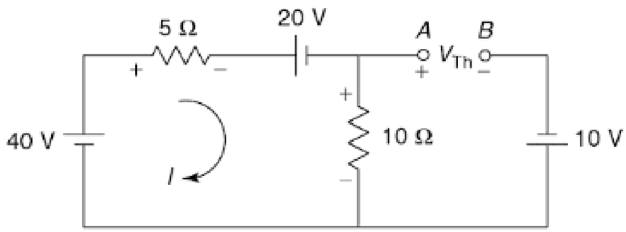


Fig. 3.116

Writing the V_{Th} equation,

$$10I - V_{Th} + 10 = 0$$

$$V_{Th} = 10I + 10 = 10(1.33) + 10 = 23.33 \text{ V}$$

Step II Calculation of R_{Th} (Fig. 3.117)

$$R_{Th} = 5 \parallel 10 = 3.33\ \Omega$$

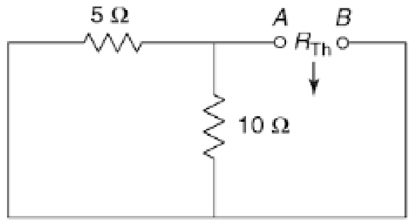


Fig. 3.117

Step III Calculation of I_L (Fig. 3.118)

$$I_L = \frac{23.33}{3.33 + 2} = 4.38 \text{ A}$$

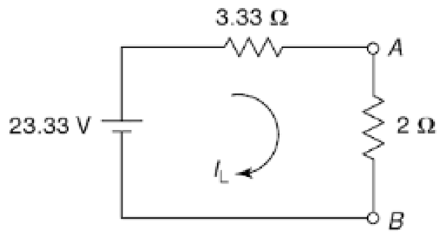


Fig. 3.118

NORTON'S THEOREM

It states that '*any two terminals of a network can be replaced by an equivalent current source and an equivalent parallel resistance.*' The constant current is equal to the current which would flow in a short circuit placed across the terminals. The parallel resistance is the resistance of the network when viewed from these open-circuited terminals after all voltage and current sources have been removed and replaced by internal resistances.

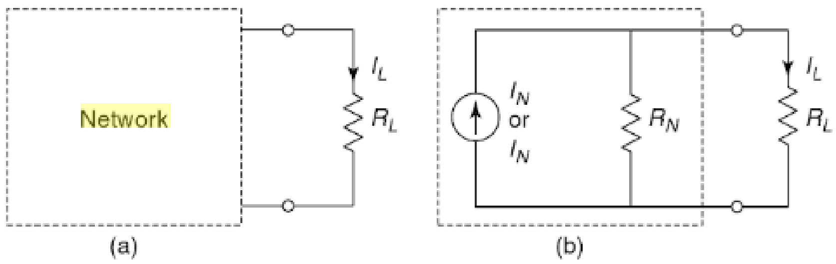


Fig. 3.251 *Network illustrating Norton's theorem*

Explanation Consider a simple network as shown in Fig.3.252

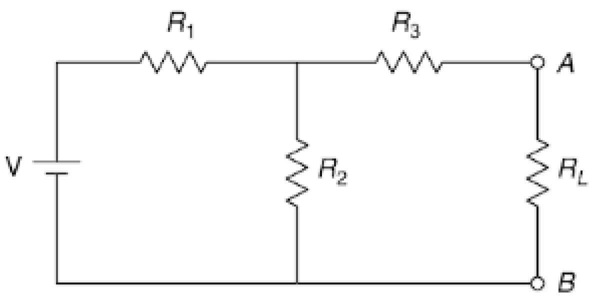


Fig. 3.252 *Network*

For finding load current through R_L, first remove the load resistor R_L from the network and calculate short circuit current I_{SC} or I_N which would flow in a short circuit placed across terminals A and B as shown in Fig. 3.253.

For finding parallel resistance R_N, replace the voltage source by a short circuit and calculate resistance between points A and B as shown in Fig. 3.254.

$$R_N = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$

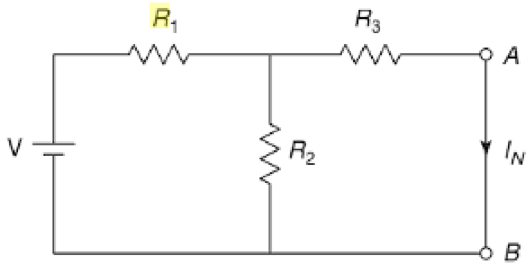


Fig. 3.253 *Calculation of I_N*

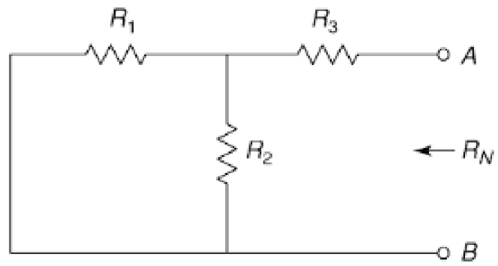


Fig. 3.254 *Calculation of R_N*

Norton's equivalent network is shown in Fig.3.255.

$$I_L = I_N \frac{R_N}{R_N + R_L}$$

If the network contains both independent and dependent sources, Norton's resistances R_N is calculated as

$$R_N = \frac{V_{Th}}{I_N}$$

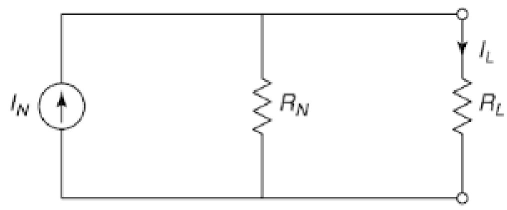


Fig. 3.255 *Norton's equivalent network*

where V_{Th} is the open-circuit voltage across terminals A and B. If the network contains only dependent sources, then

$$V_{Th} = 0$$

$$I_N = 0$$

To find R_{Th} in such network, a known voltage V or current I is applied across the terminals A and B, and the current I or the voltage V is calculated respectively.

$$R_N = \frac{V}{I}$$

Norton's equivalent network for such a network is shown in Fig. 3.256.

Fig. 3.256 *Norton's equivalent network*

Steps to be followed in Norton's Theorem

1. Remove the load resistance R_L and put a short circuit across the terminals.
2. Find the short-circuit current I_{SC} or I_N.
3. Find the resistance R_N as seen from points A and B.
4. Replace the network by a current source I_N in parallel with resistance R_N.
5. Find current through R_L by current–division rule.

$$I_L = \frac{I_N R_N}{R_N + R_L}$$

Example 3.59 *Find the current through the 10 Ω resistor in Fig. 3.258.*

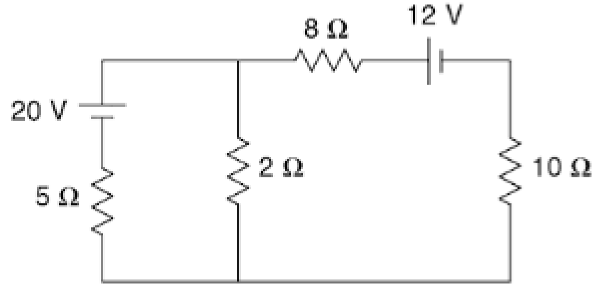


Fig. 3.261

Solution

Step I Calculation of I_N (Fig. 3.262)

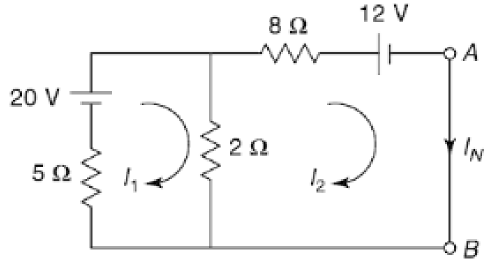


Fig. 3.262

Applying KVL to Mesh 1,

$$-5I_1 + 20 - 2(I_1 - I_2) = 0$$

$$7I_1 - 2I_2 = 20 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-2(I_2 - I_1) - 8I_2 - 12 = 0$$

$$-2I_1 + 10I_2 = -12 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii),

$$I_2 = -0.67\text{ A}$$

$$I_N = I_2 = -0.67\text{ A}$$

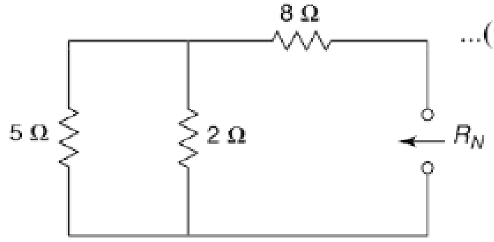


Fig. 3.263

Step II Calculation of R_N (Fig. 3.263)

$$R_N = (5 \| 2) + 8 = 9.43\ \Omega$$

Step III Calculation of I_L (Fig. 3.264)

$$I_L = 0.67 \times \frac{9.43}{9.43 + 10} = 0.33\text{ A}(\uparrow)$$

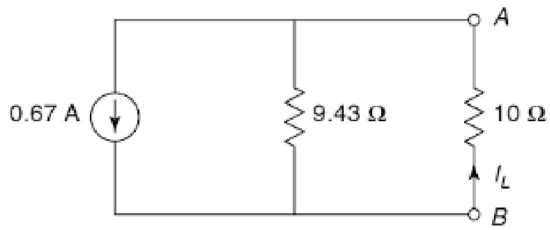


Fig. 3.264

MAXIMUM POWER TRANSFER THEOREM

It states that '*the maximum power is delivered from a source to a load when the load resistance is equal to the source resistance.*'

Proof From Fig. 3.363,

$$I = \frac{V}{R_s + R_L}$$

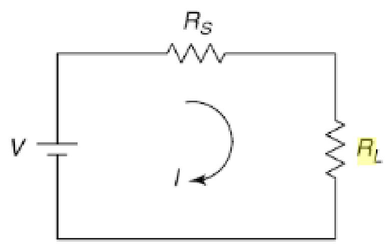


Fig. 3.363 *Network illustrating maximum power transfer theorem*

Power delivered to the load $R_L = P = I^2\ R_L = \dfrac{V^2 R_L}{(Rs + R_L)^2}$

To determine the value of R_L for maximum power to be transferred to the load,

$$\frac{dP}{dR_L} = 0$$

$$\frac{dP}{dR_L} = \frac{d}{dR_L}\frac{V^2}{(R_s + R_L)^2}R_L$$

$$= \frac{V^2[(R_s + R_L)^2 - (2R_L)(R_s + R_L)]}{(R_s + R_L)^4}$$

$$(R_s + R_L)^2 - 2\,R_L(R_s + R_L) = 0$$

$$R_s^2 + R_L^2 + 2R_sR_L - 2R_LR_s - 2R_L^2 = 0$$

$$R_s = R_L$$

Hence, the maximum power will be transferred to the load when load resistance is equal to the source resistance.

Steps to be followed in Maximum Power Transfer Theorem

1. Remove the variable load resistor R_L.
2. Find the open circuit voltage V_{Th} across points A and B.
3. Find the resistance R_{Th} as seen from points A and B.
4. Find the resistance R_L for maximum power transfer.

$$R_L = R_{Th}$$

5. Find the maximum power (Fig. 3.364).

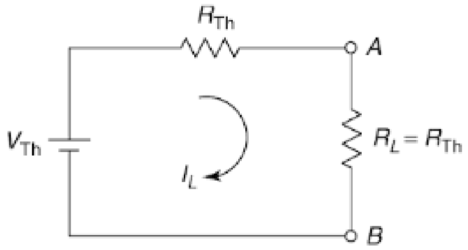


Fig. 3.364 *Thevenin's equivalent network*

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{V_{Th}}{2R_{Th}}$$

$$P_{max} = I_L^2\,R_L = \frac{V_{Th}^2}{4R_{Th}^2} \times R_{Th} = \frac{V_{Th}^2}{4R_{Th}}$$

Example 3.82 *Find the value of resistance R_L in Fig. 3.365 for maximum power transfer and calculate maximum power.*

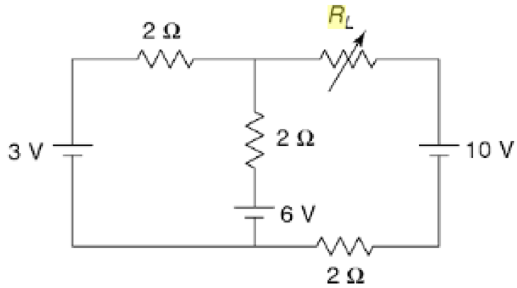


Fig. 3.365

Solution

Step I Calculation of V_{Th} (Fig. 3.366)

Applying KVL to the mesh,

$$3 - 2I - 2I - 6 = 0$$

$$I = -0.75\text{ A}$$

Writing the V_{Th} equation,

$$6 + 2I - V_{Th} - 10 = 0$$

$$V_{Th} = 6 + 2I - 10 = 6 + 2(-0.75) - 10 = -5.5\text{ V}$$

$$= 5.5\text{ V (terminal } B \text{ is positive w.r.t } A)$$

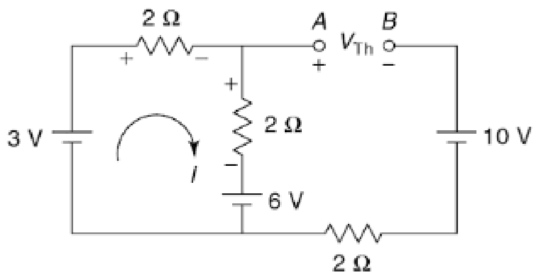


Fig. 3.366

Step II Calculation of R_{Th} (Fig. 3.367)

$$R_{Th} = (2 \,||\, 2) + 2 = 3\,\Omega$$

Step III Calculation of R_L

For maximum power transfer,

$$R_L = R_{Th} = 3\,\Omega$$

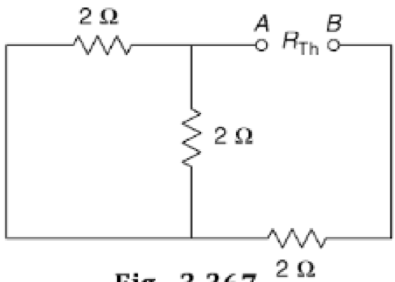


Fig. 3.367

Step IV Calculation of P_{max} (Fig. 3.368)

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(5.5)^2}{4 \times 3} = 2.52\text{ W}$$

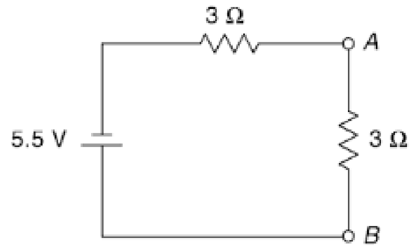


Fig. 3.368

Compensation Theorem

In circuit analysis, many times it is required to study the effect of change in impedance in one of its branches on the corresponding voltages and currents of the network. The compensation theorem provides a very simple way for studying such effects. The statement of compensation theorem is as follows.

Statement : *In any linear network consisting of linear and bilateral impedances and active sources, if the impedance Z of the branch carrying current I increases by δZ, then the increament of voltage or current in each branch of the network is that voltage or current that would be produced by an opposing voltage source of value* $V_C(=I.\delta Z)$ *introduced in the altered branch after replacing original sources by their internal impedances.*

7.6.1 Explanation of Compensation Theorem

Consider a network shown in the Fig. 7.21.

(a) (b)

Fig. 7.21

V is the voltage applied to the network. I is the current flowing through Z_1 and Z_2. Consider that impedance Z_2 increases by δZ . Due to this, the current in the circuit changes to I′ as shown in the Fig. 7.21 (b).

Then the effect of change in impedance is the change in current which is given by,

$$\delta I = I - I'$$

Now this current can be directly calculated by using the compensation theorem. First modify the branch of which impedance is changed, by connecting a voltage source V_C of value $I \cdot \delta Z$. The new voltage source must be connected in the branch with proper polarity.

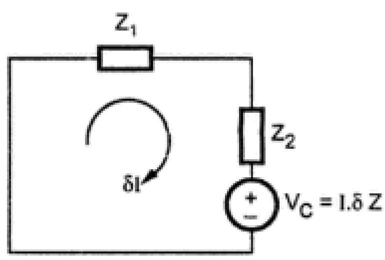


Fig. 7.22

Then replace original active source i.e. voltage source V by its internal impedance as shown in the Fig. 7.22.

The voltage source introduced in modified branch, V_C is called **compensation source** with value $I \cdot \delta Z$ where I is current through impedance before impedance of the branch is changed and δZ is the change in impedance.

Proof of Compensation Theorem

Consider a network shown in the Fig. 7.23.

(a) (b)

Fig. 7.23

The current flowing in the circuit is given by,

$$I = \frac{V}{Z_1 + Z_2} \quad \text{... (1)}$$

Consider that the impedance Z_2 changes by δZ, then the current changes to I' as shown in the Fig. 7.23 (b).

The current I' is given by,

$$I' = \frac{V}{Z_1 + (Z_2 + \delta Z)} \quad \text{... (2)}$$

The change in current due to change in impedance is given by,

$$\delta I = I - I'$$

$$= \frac{V}{Z_1 + Z_2} - \frac{V}{Z_1 + (Z_2 + \delta Z)} = \frac{V}{(Z_1 + Z_2)} - \frac{V}{(Z_1 + Z_2 + \delta Z)}$$

$$= \frac{V(Z_1 + Z_2 + \delta Z) - V(Z_1 + Z_2)}{(Z_1 + Z_2)(Z_1 + Z_2 + \delta Z)}$$

$$= \frac{V[Z_1 + Z_2 + \delta Z - Z_1 - Z_2]}{(Z_1 + Z_2)(Z_1 + Z_2 + \delta Z)}$$

$$= \frac{V}{(Z_1 + Z_2)} \cdot \frac{\delta Z}{(Z_1 + Z_2 + \delta Z)}$$

$$= \frac{I \cdot \delta Z}{(Z_1 + Z_2 + \delta Z)} \quad \text{... from equation (1)}$$

$$\therefore \quad \delta I = \frac{V_C}{(Z_1 + Z_2 + \delta Z)} \quad \text{... (3)}$$

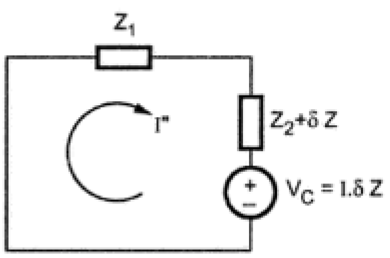


Fig. 7.24

Now consider that the branch is modified as shown in the Fig. 7.24 and also original voltage source is short circuited. Let the current in circuit be I".

Applying KVL to the loop,

$$-Z_1 \cdot I'' - (Z_2 + \delta Z) \cdot I'' + V_C = 0$$

$$\therefore \quad -(Z_1 + Z_2 + \delta Z) I'' = -I \cdot \delta Z$$

$$\therefore \quad I'' = \frac{I \cdot \delta Z}{(Z_1 + Z_2 + \delta Z)} = \frac{V_C}{(Z_1 + Z_2 + \delta Z)} \quad \text{... (4)}$$

From equations (3) and (4), $I'' = \delta I$

Thus, compensation theorem is proved.

Example 7.7 : *Calculate change in current in the network shown in the Fig. 7.25 by using compensation theorem when the reactance has changed to j35 Ω.*

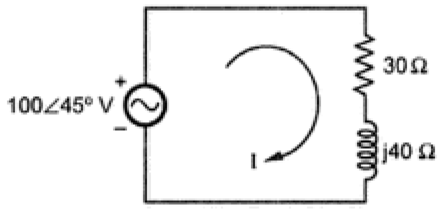


Fig. 7.25

Solution : Applying KVL, we get,

$$I = \frac{100\angle 45°}{30+j40} = \frac{100\angle 45°}{50\angle 53.13°} = 2\angle -8.13° \text{ A}$$

$$\therefore \quad I = (1.9798 - j0.2828) \text{ A} \qquad \dots (1)$$

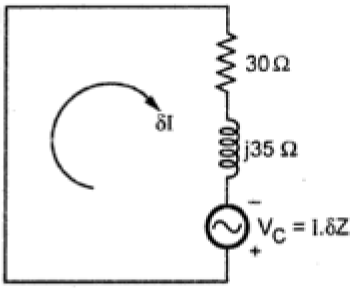


Fig. 7.25 (a)

Now the reactance has changed to j35. Hence the current in network will also change to I′. The change in the reactance is given by,

$$\delta Z = j40 - j35 = j5\ \Omega \qquad \dots (2)$$

Now the reactance is decreased. Modifying the network by replacing voltage source by short circuit and introducing compensation source $V_C = I \cdot \delta Z$ in the branch altered as shown in the Fig. 7.25(a).

The compensation source is given by,

$$V_C = I \cdot \delta Z$$

$$= (2\angle -8.13°)(j5) = (2\angle -8.13°)(5\angle 90°)$$

$$\therefore \quad V_C = 10\angle 81.87° \text{ V} \qquad \dots (3)$$

Thus, change in current is given by,

$$\delta I = \frac{V_C}{30+j35} = \frac{10\angle 81.87°}{46.0977\angle 49.4°} = \mathbf{0.2169\angle 32.47° \text{ A}}$$

Substitution Theorem

In network analysis many times it is needed to replace an impedance branch by another branch with different network elements without disturbing the voltages and currents in the network. The substitution theorem provides the convenient method to get the condition under which branch replacement is possible. The statement of substitution theorem is as below.

Statement : *In any network any branch of it may be replaced (substituted) by a branch with different network elements without disturbing the voltages and currents in the entire network, if the new branch has same set of terminal voltage and current as the original branch.*

Explanation of Substitution Theorem

Consider a network shown in the Fig. 7.26. Let the current through branch AB be I_{AB} and voltage across the branch A-B be V_{AB}.

The voltage across original branch AB is given by,

$$V_{AB} = Z_{AB} \cdot I_{AB} + E \qquad \dots (1)$$

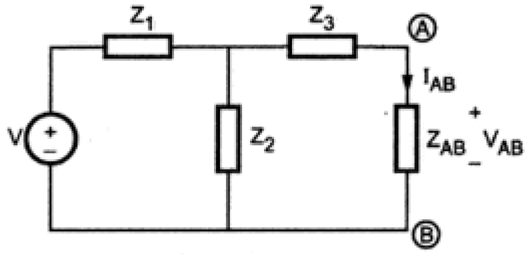


Fig. 7.26

This branch may be substituted by any other branch in many other ways where the branch voltage is given by

$$V_{AB} = Z'_{AB} I_{AB} + E' \quad \text{... (2)}$$

where Z'_{AB} and E' are chosen such that I_{AB} and V_{AB} are not changed.

Following are the important points in the accordance with the application of the substitution theorem :

1) The substitution theorem is applicable to both the types of the networks such as linear and nonlinear.
2) If the substitution theorem is applied in non-linear network then a modified network must have a unique solution. In linear networks it has number of solutions.
3) The substitution theorem is useful in proving other network theorems.
4) The substitution theorem is useful in analysis of a network having one non-linear element.

Example 7.8 : *For the network shown in the Fig. 7.27, substitute the branch A-B by a) a voltage source b) a current source.*

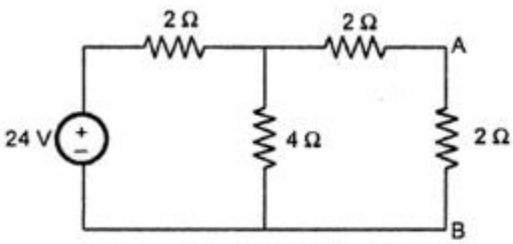


Fig. 7.27

Solution : Let current through branch A-B be I_{AB}.

Total resistance looking from source is given by

$$R_{eq} = 2+[4||(2+2)] = 2+[4||4] = 4\ \Omega$$

$$\therefore \text{ Total current } \quad I_T = \frac{24}{4} = 6\,A$$

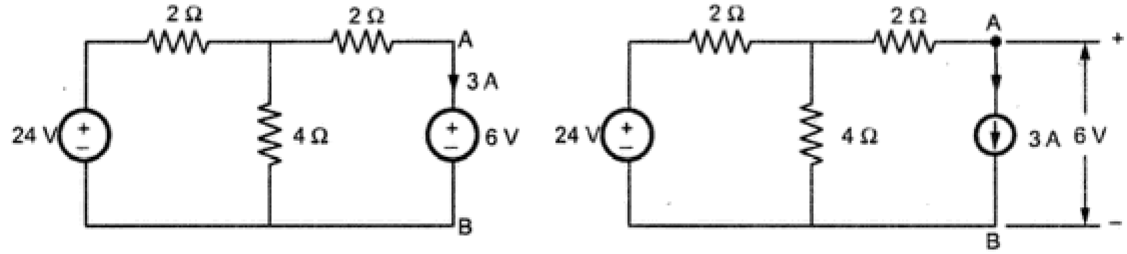
By current divider rule,

$$I_{AB} = 6\left[\frac{4}{4+4}\right] = 3\,A$$

∴ Voltage across branch A-B,

$$V_{AB} = I_{AB} \times 2 = 3 \times 2 = 6\ V$$

Therefore branch A-B may be substituted either by an independent voltage source of value 6 V with the branch current of 3 A or by an independent current source of value 3 A with the branch voltage of 6 V. The two substitutions are as shown in the Fig. 7.27 (a) and (b) and respectively.



(a) (b)

Fig. 7.27

Millman's Theorem

It is possible to combine number of voltage sources or current sources into a single equivalent voltage or current source, using Millman's theorem. The statement of the Millman's theorem is,

Statement : *If n voltage sources $V_1, V_2, ..., V_n$ having internal impedances (or series impedances) $Z_1, Z_2, ..., Z_n$ respectively, are in parallel, then these sources may be replaced by a single voltage source of voltage V_M having a series impedance Z_M where V_M and Z_M are given by,*

$$V_M = \frac{V_1 Y_1 + V_2 Y_2 + ...+ V_n Y_n}{Y_1 + Y_2 + ...+ Y_n} = \frac{\sum_{k=1}^{n} V_k Y_k}{\sum_{k=1}^{n} Y_k}$$

and

$$Z_M = \frac{1}{Y_1 + Y_2 + ...+ Y_n} = \frac{1}{\sum_{k=1}^{n} Y_k}$$

where $Y_1, Y_2, ..., Y_n$ are the admittances corresponding to the impedances $Z_1, Z_2, ..., Z_n$.

Proof of Millman's Theorem

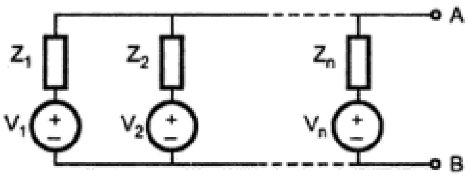


Fig. 7.33

Consider n voltage sources in parallel as shown in the Fig. 7.33.

Let us convert each voltage source into an equivalent current source. For source 1,

$$I_1 = \frac{V_1}{Z_1} = V_1\ Y_1 \text{ as } Y_1 = \frac{1}{Z_1}$$

Similarly for the remaining sources we can write,

$$I_2 = V_2\ Y_2\ ,\quad I_3 = V_3\ Y_3, ...I_n = V_n Y_n$$

where $Y_1, Y_2, ...Y_n$ are the admittances to be connected in parallel. Hence circuit reduces to,

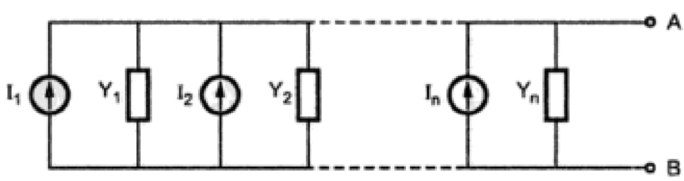


Fig. 7.33 (a)

Hence the effective current source across the terminals A-B is,

$$I_M = I_1 + I_2 + \ldots\ldots\ldots + I_n \quad \ldots (1)$$

and
$$Y_M = Y_1 + Y_2 + \ldots\ldots\ldots + Y_n \quad \ldots (2)$$

This is because admittances in parallel get added to each other. Hence circuit reduces to, as shown in the Fig. 7.34.

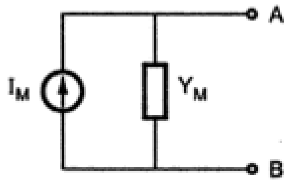


Fig. 7.34

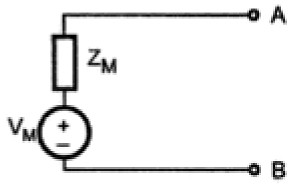


Fig. 7.34 (a)

Converting this equivalent current source into the voltage source we get,

$$V_M = \frac{I_M}{Y_M}$$

as
$$Z_M = \frac{1}{Y_M}$$

$$V_M = I_M Z_M$$

Substituting I_M and Y_M from equations (1) and (2),

$$V_M = (I_1 + I_2 + \ldots\ldots\ldots + I_n) \cdot \frac{1}{(Y_1 + Y_2 + \ldots + Y_n)}$$

but
$$I_1 = \frac{V_1}{Z_1} = V_1 Y_1, \; I_2 = V_2 Y_2, \ldots I_n = V_n Y_n$$

$$\therefore \quad \boxed{V_M = \frac{V_1 Y_1 + V_2 Y_2 + \ldots + V_n Y_n}{Y_1 + Y_2 + \ldots + Y_n}}$$

$$\therefore \quad \boxed{Z_M = \frac{1}{Y_M} = \frac{1}{Y_1 + Y_2 + \ldots + Y_n}}$$

Thus Millman's theorem is proved.

Example 7.10 : *Use Millman's theorem to find the current through the 10 Ω resistance in the circuit of Fig. 7.35.*

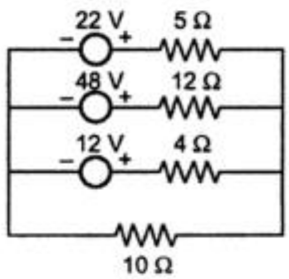


Fig. 7.35

Solution : From the given network we can write,

$$V_1 = 12 \text{ V}, Z_1 = 4 \,\Omega, V_2 = 48 \text{ V}, Z_2 = 12 \,\Omega, V_3 = 22 \text{ V}, Z_3 = 5 \,\Omega$$

$$\therefore \quad Y_1 = \frac{1}{4} \text{ mho}, Y_2 = \frac{1}{12} \text{ mho}, Y_3 = \frac{1}{5} \text{ mho}$$

According to Millman's theorem, the equivalent voltage source and impedance across 10 Ω is given by,

$$Z_M = \frac{1}{Y_1 + Y_2 + Y_3} = \frac{1}{\frac{1}{4} + \frac{1}{12} + \frac{1}{5}}$$

$$= 1.875 \,\Omega$$

$$V_M = \frac{V_1Y_1 + V_2Y_2 + V_3Y_3}{Y_1 + Y_2 + Y_3} = \frac{12\times\frac{1}{4}+48\times\frac{1}{12}+22\times\frac{1}{5}}{05333}$$

$$= \mathbf{21.375\ V}$$

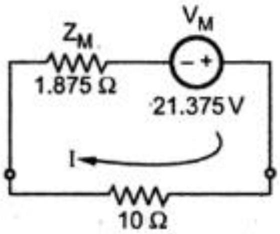


Fig. 7.35 (a)

The equivalent is shown in the Fig. 7.35 (a).

$$I_{10\Omega} = \frac{V_M}{Z_M + 10}$$

$$= \frac{21.375}{1.875+10} = \mathbf{1.8\ A}$$

Tellegan's Theorem

The Tellegan's theorem is valid for any lumped network which may be linear or nonlinear, active or passive, time varying or time invariant. The statement of the theorem is as below :

Statement : *In an arbitrary lumped network, the algebraic sum of the instantaneous powers in all the branches, at any instant is zero. All the branch currents and the voltages in that network must satisfy Kirchhoff's laws. In other words, it can be stated as, in a given network, the algebraic sum of the powers delivered by all the sources is equal to the algebraic sum of the powers absorbed by all the elements.*

Mathematically this theorem can be expressed as,

$$\sum_{k=1}^{b} v_k\, i_k = 0$$

where b is the number of branches in a network.

Explanation of Tellegan's Theorem

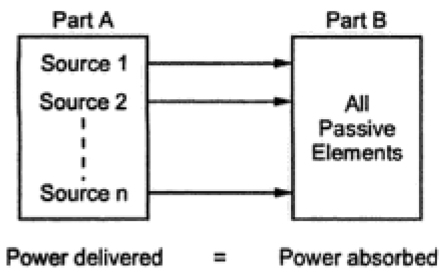


Fig. 7.36

Let the network is divided into two parts. The part A with 'n' active energy sources and second part B with all the passive elements. Then the power delivered by n sources of part A must be equal to the sum of the power absorbed (dissipated or stored) by the elements of part B. This is shown in the Fig. 7.36.

Example 7.11 : *For the network shown in the Fig. 7.37 verify Tellegan's theorem.*

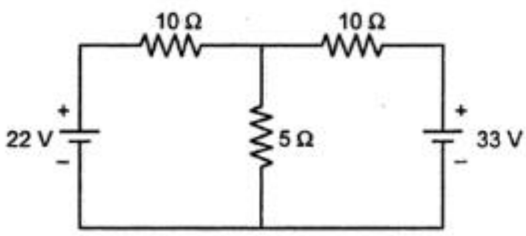


Fig. 7.37

Solution : Assuming loop currents as shown in the Fig. 7.37 (a).

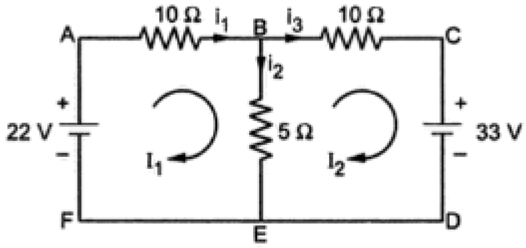


Fig. 7.37 (a)

Applying KVL to loop A-B-E-F-A,

$$-10I_1 - 5I_1 + 5I_2 + 22 = 0$$

$$\therefore \quad -15I_1 + 5I_2 = -22$$

$$\therefore \quad 15I_1 - 5I_2 = 22 \quad \text{... (1)}$$

Applying KVL to loop B-C-D-E-B,

$$-10I_2 - 33 - 5I_2 + 5I_1 = 0$$

$$\therefore \quad 5I_1 - 15I_2 = 33 \quad \text{... (2)}$$

Solving equations (1) and (2) simultaneously,

$$I_1 = 0.825 \text{ A}$$

$$I_2 = -1.925 \text{ A}$$

Current flowing through loop2 is negative which indicates that assumed direction of I_2 is exactly opposite to the actual direction. Hence I_2 flows in anticlockwise direction. $\therefore I_2 = 1.925$ A in anticlockwise direction.

Current through 10 Ω resistor between nodes A and B is given by,

$$i_1 = I_1 = 0.825 \text{ A....... from A to B}$$

Current flowing through 5 Ω resistor is given by

$$i_2 = I_1 + I_2 = (0.825) + (0.1925) = 2.75 \text{ A} \quad \text{....... From B to E}$$

Current flowing through 10 Ω resistor between nodes B and C is given by,

$$i_3 = I_2 = 1.925 \text{ A ...... from C to B.}$$

Total power delivered by sources,

$$P_{delivered} = (I_1)(22) + (I_2)(33)$$

$$= (0.825)(22) + (1.925)(33)$$

$$= \mathbf{81.675 \text{ W}}$$

Total power absorbed by the elements

$$P_{absorbed} = (i_1^2 \times 10) + (i_2^2 \times 5) + (i_3^2 \times 10)$$

$$= [(0.825)^2 \times 10] + [(2.75)^2 \times 5] + [(1.925)^2 \times 10]$$

$$= \mathbf{81.675 \text{ W}}$$

$$\therefore \quad P_{delivered} = P_{absorbed}$$ ..... Hence Tellegan's theorem is proved.

Reciprocity Theorem

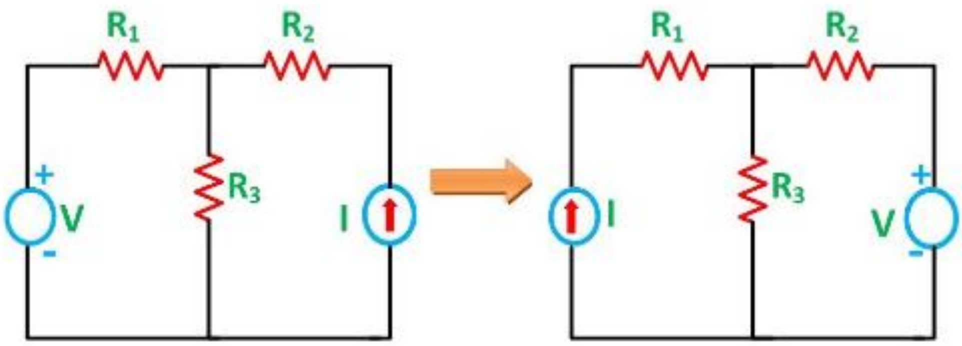
Reciprocity Theorem states that – In any branch of a network or circuit, the current due to a single source of voltage (V) in the network is equal to the current through that branch in which the source was originally placed when the source is again put in the branch in which the current was originally obtained.This theorem is used in the bilateral linear network which consists bilateral components.

In simple words, we can state the reciprocity theorem as when the places of voltage and current source in any network are interchanged the amount or magnitude of current and voltage flowing in the circuit remains the same. This theorem is used for solving many DC and AC network which have many applications in electromagnetism electronics.Their circuit does not have any time varying element.

Explanation of Reciprocity Theorem

The location of the voltage source and the current source may be interchanged without a change in current. However, the polarity of the voltage source should be identical with the direction of the branch current in each position.

The Reciprocity Theorem is explained with the help of the circuit diagram shown below



The various resistances R_1, R_2, R_3 is connected in the circuit diagram above with a voltage source (V) and a current source (I). It is clear from the figure above that the voltage source and current sources are interchanged for solving the network with the help of Reciprocity Theorem.

The limitation of this theorem is that it is applicable only to single source networks and not in the multi-source network. The network where reciprocity theorem is applied should be linear and consist of resistors, inductors, capacitors and coupled circuits. The circuit should not have any time-varying elements.

Steps for Solving a Network Utilizing Reciprocity Theorem

Step 1 – Firstly, select the branches between which reciprocity has to be established.

Step 2 – The current in the branch is obtained using any conventional network analysis method.

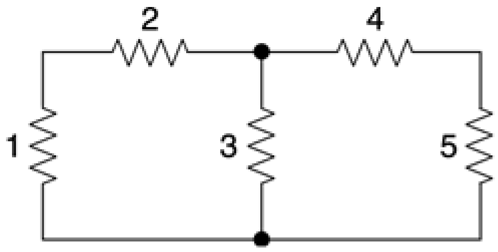
Step 3 – The voltage source is interchanged between the branch which is selected.

Step 4 – The current in the branch where the voltage source was existing earlier is calculated.

Step 5 – Now, it is seen that the current obtained in the previous connection, i.e., in step 2 and the current which is calculated when the source is interchanged, i.e., in step 4 are identical to each other.

Problems-14

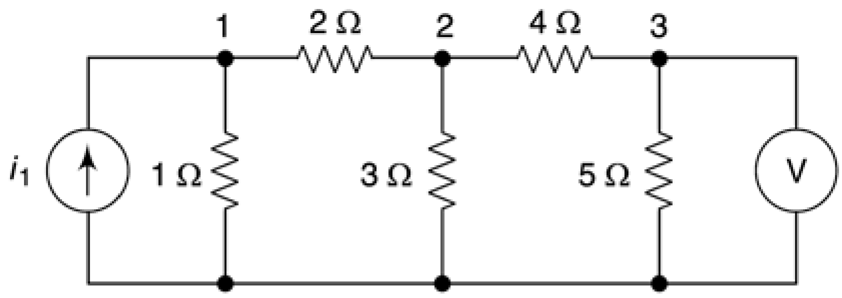
Verify the Reciprocity Theorem for the network shown in the figure using current source and a voltmeter. All the values are in ohm.



Solution

Using a current source and a voltmeter,

Let, e_1, e_2 be node voltages, v_1 be the voltmeter reading.



By KCL,

At node (1) $\Rightarrow 3e_1 - e_2 - 2i_1 = 0$ (i)

At node (2) $\Rightarrow -6e_1 + 13e_2 - 3v_1 = 0$ (ii)

At node (3) $9v_1 = 5e_2$ (iii)

From (ii) $\Rightarrow -6e_1 + 13 \times \frac{9}{5} v_1 - 3v_1 = 0$

$\Rightarrow -6e_1 + \left(\frac{117}{5} - 3\right) v_1 = 0$

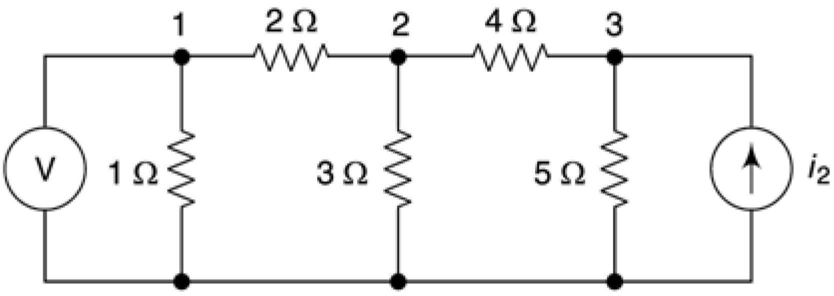
$\Rightarrow 6e_1 + \frac{102}{5} v_1 \Rightarrow e_1 = \frac{17}{5} v_1$

From (i) $\Rightarrow 3 \times \frac{17}{5} v_1 - \frac{9}{5} v_1 = 2i$

$\Rightarrow \left(\frac{i_1}{v_1}\right) = \left(\frac{21}{5}\right)$ (A)

Interchanging the positions of the current source and the voltmeter,

Now, let v_2 be the voltmeter reading



By KCL,

At node (1) $\Rightarrow 3v_2 = e_2$ (iv)

At node (2) $\Rightarrow -6v_2 + 13e_2 - 3e_3 = 0$

$\Rightarrow -6v_2 + 13 \times 3v_2 - 3e_3 = 0$

$\Rightarrow e_3 = 11v_2$ (v)

At node (3) $\Rightarrow 5e_3 - 5e_2 + 4e_3 - 20i_2 = 0$

$\Rightarrow 20i_2 = 9e_3 - 5e_2 = 9 \times 11v_2 - 5 \times 3v_2 = 84v_2$

$\Rightarrow \left(\frac{i_2}{v_2}\right) = \left(\frac{21}{5}\right)$ (B)

From equations (A) and (B), Reciprocity theorem is proved.